1. a) (4 points) An LU-factorization for a 4 × 4 matrix $A$ is given as

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \]

Evaluate the determinant of $A$. Is $A$ invertible?
b) (3 points) Evaluate the determinant of the matrix
\[ B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}. \]

2. (Each part is 1 point) Determine whether each of the following statements is true or false. For each part circle either T(true) or F(false). You do not need to justify your answers.

(i) An \( n \times n \) matrix \( A \) is diagonalizable if and only if \( A \) has \( n \) eigenvectors that are mutually orthogonal. \[ \underline{T} \quad \underline{F} \]

(ii) The orthogonal projection of a vector \( v \) onto a subspace \( W \) yields the component \( \hat{v} \) of \( v \) contained on the subspace \( W \) such that \( (v - \hat{v}) \) is orthogonal to \( \hat{v} \). \[ \underline{T} \quad \underline{F} \]

(iii) The set of polynomials is an infinite dimensional vector space. \[ \underline{T} \quad \underline{F} \]

(iv) Given an \( n \times n \) matrix \( A \), if the linear system \( Ax = b \) is inconsistent for some \( b \), then \( A \) is not invertible. \[ \underline{T} \quad \underline{F} \]

(v) The set \( \{ c_1 c_2 + c_1 x + c_2 x^2 : c_1, c_2 \in \mathbb{R}^2 \} \) is a two dimensional subspace of the vector space of polynomials of degree at most two \( \mathbb{P}_2 = \{ c_0 + c_1 x + c_2 x^2 : c_0, c_1, c_2 \in \mathbb{R} \} \). \[ \underline{T} \quad \underline{F} \]

(vi) If the rank of a \( 5 \times 7 \) matrix \( A \) is 4, then the null space of \( A \) is a 1-dimensional subspace of \( \mathbb{R}^5 \). \[ \underline{T} \quad \underline{F} \]

(vii) Let \( v \) be a vector in an \( n \)-dimensional vector space \( V \) spanned by the basis \( B = \{ b_1, b_2, \ldots, b_n \} \). If \( v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n = \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_n b_n \) for some \( \alpha_1, \alpha_2, \beta_1, \beta_2, \ldots, \alpha_n, \beta_n \in \mathbb{R} \), then \( \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n \), that is there is a unique way \( v \) can be written as a linear combination of \( b_1, b_2, \ldots, b_n \). \[ \underline{T} \quad \underline{F} \]
3. Suppose a $QR$-factorization of $A$ is given by

\[
A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 0 & 1 \\ 4 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -4 & 0 \\ \frac{1}{5} & 0 & 0 & 5 \\ 4 & 4 & 3 & 0 \end{pmatrix} \begin{pmatrix} 5 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}
\]

where $Q$ is an orthogonal matrix (that is $Q^TQ = I$, therefore $Q^T$ is the inverse of $Q$).

a) (4 points) Using the $QR$-factorization above find the inverse of $A$.

b) (3 points) Using the $QR$-factorization solve the linear system

\[
Ax = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]
4. Let
\[ A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]

a) (3 points) Find a basis for the column space of \( A \). What is the rank of \( A \)? Is \( A \) invertible?

b) (4 points) Find all least squares solutions \( \hat{x} \) satisfying
\[ \| b - A\hat{x} \| \leq \| b - Ax \| \quad \text{for all} \quad x \in \mathbb{R}^3. \]
5. Suppose $A$ is a $3 \times 3$ matrix with the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 1$ and the eigenvectors
\[ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]
corresponding to $\lambda_1$, $\lambda_2$ and $\lambda_3$, respectively. Consider also the dynamical system
\[ x_k = Ax_{k-1} \]
with the initial condition
\[ x_0 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \]

a) (1 point) Write $x_0$ as a linear combination of $v_1$, $v_2$ and $v_3$, that is determine the scalars $\alpha_1$, $\alpha_2$ and $\alpha_3$ such that
\[ x_0 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3. \]

b) (4 points) Using the expression for $x_0$ as a linear combination of the eigenvectors you determined in part a) find $x_3 = A^3 x_0$. 
c) **(3 points)** What are the eigenvalues and the corresponding eigenvectors of $A - 2I$.

6. Let $\mathbb{P}_2 = \{c_1 + c_2x + c_3x^2 : c_1, c_2, c_3 \in \mathbb{R}\}$ be the vector space of polynomials of degree at most two. Define the linear transformation $T : \mathbb{P}_2 \rightarrow \mathbb{R}^2$ as

$$T(c_1 + c_2x + c_3x^2) = \begin{bmatrix} c_1 \\ c_2 + c_3 \end{bmatrix}.$$

a) **(3 points)** Find a basis for the kernel of $T$. Is $T$ one-to-one?

b) **(3 points)** Find a basis for the range of $T$. Is $T$ onto $\mathbb{R}^2$?
7. Let 
\[ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \]

a) (2 points) Is the set \( \{v_1, v_2, v_3\} \) linearly independent?

b) (2 points) Is the set \( \{v_1, v_2, v_3\} \) orthogonal?

c) (3 points) Find an orthogonal basis for the subspace \( \mathcal{W} = \text{span}\{v_1, v_2, v_3\} \).
8. (6 points) Find the eigenvalues and the corresponding eigenvectors for the matrix

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}. \]

Indicate which eigenvector corresponds to which eigenvalue. Is \( A \) diagonalizable?