WEEK 12

FINITE DIFFERENCE METHODS
FOR PARTIAL DIFFERENTIAL EQUATIONS

Finite Difference Formulas

\[ f : \mathbb{R} \to \mathbb{R}, \text{ differentiable} \]

\[
\frac{f'(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

1. Forward Difference

Fix \( h > 0 \)

\[ D_{f}f(x) := \frac{f(x+h) - f(x)}{h} \]

For \( h \approx 0 \)

\[ f'(x) \approx D_{f}f(x) \]
2 Backward Difference

Fix \( h > 0 \), and let \( \tilde{h} := -h \)

\[
D_B f(x) := \frac{f(x + \tilde{h}) - f(x)}{\tilde{h}} = \frac{f(x - h) - f(x)}{-h} \approx \frac{f(x) - f(x-h)}{h}
\]

For \( \tilde{h} \approx 0 \) we also have \( f'(x) \approx D_B f(x) \)

![Graph showing the tangent line and the backward difference](image)
3 Central Difference

Fix $h > 0$

$$D_c f(x) := \frac{D_F f(x) + D_B f(x)}{2} = \frac{\left[\frac{f(x+h) - f(x)}{h}\right] + \left[\frac{f(x) - f(x-h)}{h}\right]}{2} = \frac{f(x+h) - f(x-h)}{2h}$$

Example

$$f(x) = e^x$$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$D_F f(0)$</th>
<th>$D_B f(0)$</th>
<th>$D_c f(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0517</td>
<td>0.9516</td>
<td>1.00166750</td>
</tr>
<tr>
<td>0.01</td>
<td>1.0050</td>
<td>0.9950</td>
<td>1.00001667</td>
</tr>
<tr>
<td>0.001</td>
<td>1.0005</td>
<td>0.9995</td>
<td>1.0000017</td>
</tr>
</tbody>
</table>

$f'(0) - D_F f(0)$ proportional to $h$

$f'(0) - D_B f(0)$ proportional to $h$

$f'(0) - D_c f(0)$ proportional to $h^2$
Accuracy

\[ f'(x) - D_f f(x) = f'(x) - \frac{f(x+h) - f(x)}{h} \]

\[ = f'(x) - \left[ \frac{f(x) + f'(x)h + f''(\xi)(\frac{h^2}{2}) - f(x)}{h} \right] \]

\[ = - f''(\xi) \frac{h}{2} = O(h) \]

Apply Taylor's thm about \( x \)

\[ f'(x) - D_B f(x) = f'(x) - \frac{f(x) - f(x-h)}{h} \]

\[ = f'(x) - \left[ \frac{f(x) - \{ f(x) + f'(x)(-h) + f''(\xi)(\frac{h^2}{2}) \}}{h} \right] \]

\[ = f''(\xi) \frac{h}{2} = O(h) \]

Apply Taylor's thm about \( x \)

\[ f'(x) - D_c f(x) = f'(x) - \frac{f(x+h) - f(x-h)}{2h} \]

\[ = f'(x) - \left[ \frac{\{ f(x) + f'(x)h + f''(\xi)(\frac{h^2}{2}) + f'''(\xi_1)(\frac{h^3}{6}) \} - \{ f(x) + f'(x)h + f''(\xi)(\frac{h^2}{2}) + f'''(\xi_2)(\frac{h^3}{6}) \}}{2h} \right] / (2h) \]

\[ = - \left[ \frac{f''(\xi_1) - f''(\xi_2)}{12} \right] h^2 = O(h^2) \]
Approximating Second Derivatives

1. Apply forward difference twice

\[ f''(x) \approx D_F \left[ D_F f(x) \right] \]

\[ = D_F \left[ \frac{f(x+h) - f(x)}{h} \right] \]

\[ = \frac{D_F f(x+h) - D_F f(x)}{h} \]

\[ = \frac{[f(x+2h) - f(x+h)]/h - [f(x+h) - f(x)]/h}{h} \]

\[ = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \]

2. Apply central difference twice (with step-size \( h/2 \))

\[ f''(x) \approx D_c \left[ D_c f(x) \right] \]

\[ = D_c \left[ \frac{f(x+h/2) - f(x-h/2)}{h} \right] \]

\[ = \frac{D_c f(x+h/2) - D_c f(x-h/2)}{h} \]
\[
\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}
\]

Example

\[f(x) = \frac{\cos x}{\ln x}\]

\[
\begin{array}{c|c|c}
  h & D_F^1 D_F^1 f(x) & D_C^1 D_C^1 f(x) \\
  \hline
  0.1 & -0.8299 & -1.005033585 \\
  0.01 & -0.9803 & -1.000050003 \\
  0.001 & -0.9980 & -1.000000500 \\
\end{array}
\]

Error proportional to \( h \)

Accuracy

\[
f''(x) - D_C^1 [D_C^1 f(x)] = f''(x) - \frac{[f(x+h) - 2f(x) + f(x-h)]}{h^2}
\]

\[
= f''(x) - \frac{\left\{ f(x) + f'(x)h + f''(x)(h^2/2) + f'''(x)(h^3/6) + f^{(4)}(x)(h^4/24) \right\}}{h^2}
\]

\[
= -\left[ f^{(4)}(x_1) - f^{(4)}(x_2) \right] \frac{h^2}{24} = O(h^2)
\]
Exercise

Show that

\[ f''(x) - D_x[D_x f(x)] = O(h). \]

Poisson Equation
(An elliptic PDE)

\[ u(x, y) : \mathbb{R}^2 \to \mathbb{R} \]

\[
\begin{cases}
\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = f(x, y) & \text{given} \\
\text{for all } (x, y) \in \mathbb{R} \\
\partial u(x, y) = g(x, y) & \text{for all } (x, y) \in \partial \mathbb{R}
\end{cases}
\]

where

\[ \mathbb{R} = \{ (x, y) \in \mathbb{R}^2 \mid a < x < b, \ c < y < d \} \]

\[ \partial \mathbb{R} = \text{Boundary of } \mathbb{R} \]
Examples

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2+2)e^y \quad 0 < x < 2 \]
\[ 0 < y < 1 \]
\[ u(0, y) = 0 \quad \text{and} \quad u(2, y) = 4e^y \quad 0 < y < 1 \]
\[ u(x, 0) = x^2 \quad \text{and} \quad u(x, 1) = e \cdot x^2 \quad 0 < x < 2 \]

Exact soln: \( u(x, y) = x^2e^y \)

But exact soln is usually not known for many \( f(x, y) \).

2. In physics, potential energy is typically governed by a Poisson eqn. given a density function.

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -4\pi G \cdot \rho(x, y) \]

\( \rho \) is given density function of an object

\( \phi \): (gravitational) potential energy of the object

\( \nabla \phi(x, y) \): gravitational force field exerted by the object
Numerical Solution

Discretize $\mathcal{R}$

\[
\begin{array}{c}
(a, c) \\
\bigoplus \\
(b, c) \\
(b, d) \\
(a, d)
\end{array}
\]

For simplicity suppose $(b-a)$, $(d-c)$ are divisible by $h$

\[
\begin{align*}
x_j &= a + jh, \quad j = 0, 1, 2, \ldots, \frac{b-a}{h} \\
y_j &= c + jh, \quad j = 0, 1, 2, \ldots, \frac{d-c}{h}
\end{align*}
\]

Apply central difference formula at each node (inside $\mathcal{R}$, not on $\partial \mathcal{R}$)

\[
\frac{\partial^2 u(x_j, y_k)}{\partial x^2} + \frac{\partial^2 u(x_j, y_k)}{\partial y^2} = f(x_j, y_k)
\]

\[
\left[ u(x_{j+1}, y_k) - 2u(x_j, y_k) + u(x_{j-1}, y_k) \right] / h^2
\]

\[+
\left[ u(x_j, y_{k+1}) - 2u(x_j, y_k) + u(x_j, y_{k-1}) \right] / h^2
\]

\[= f(x_j, y_k) + O(h^2)
\]
Replace \( u(x_j, y_k) \) with \( u_{j,k} \); discard \( O(h^2) \) terms.

\[
\frac{u_{j+1,k} + u_{j,k+1} - 4u_{j,k} + u_{j-1,k} + u_{j,k-1}}{h^2} = f_{j,k}
\]

where \( f_{j,k} = f(x_j, y_k) \)

This results in a linear system

\[
\begin{align*}
\text{unknowns:} & \quad u_{j,k} & j = 1, \ldots, m-1 \quad \text{(m-1)x(n-1)} \\
& & k = 1, \ldots, n-1 \\
\text{linear equations:} & \quad (*) & j = 1, \ldots, m-1 \quad \text{(m-1)x(n-1)} \\
& & k = 1, \ldots, n-1
\end{align*}
\]
Example

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xy \quad 0 < x < 3$$
$$1 < y < 4$$

$$u(0, y) = 0 \quad \text{and} \quad u(3, y) = 4y \quad 1 < y < 4$$
$$u(x, 1) = x^2 \quad \text{and} \quad u(x, 4) = 2x^2 \quad 0 < x < 3$$

Solve using central differences with $h=1$.

![Grid diagram with labels and arrows indicating unknowns and approximation formulas.]

unknowns:

$$u_{11} \approx u(h, 1+h) = u(1, 2)$$
$$u_{12} \approx u(h, 1+2h) = u(1, 3)$$
$$u_{21} \approx u(2h, 1+h) = u(2, 2)$$
$$u_{22} \approx u(2h, 1+2h) = u(2, 3)$$
linear systems

\[
\begin{align*}
\mathbf{u}_{11} &= \frac{u_{12} + u_{21} - 4u_{11} + u_{10} + u_{01}}{h^2} = h(1+h) \\
\mathbf{u}_{12} &= \frac{u_{22} + u_{13} - 4u_{12} + u_{02} + u_{11}}{h^2} = h(1+2h) \\
\mathbf{u}_{21} &= \frac{u_{31} + u_{22} - 4u_{21} + u_{11} + u_{20}}{h^2} = (2h)(1+h) \\
\mathbf{u}_{22} &= \frac{u_{32} + u_{23} - 4u_{22} + u_{12} + u_{21}}{h^2} = (2h)(1+2h)
\end{align*}
\]

that is

\[
\begin{align*}
u_{12} + u_{21} - 4u_{11} &= 1 \\
u_{22} - 4u_{12} + u_{11} &= 1 \\
u_{22} - 4u_{21} + u_{11} &= -8 \\
-4u_{22} + u_{12} + u_{21} &= -14
\end{align*}
\]

\[
\begin{bmatrix}
-4 & 1 & 1 & 0 \\
1 & -4 & 0 & 1 \\
1 & 0 & -4 & 1 \\
0 & 1 & 1 & -4
\end{bmatrix}
\begin{bmatrix}
u_{11} \\
u_{12} \\
u_{21} \\
u_{22}
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
-8 \\
-14
\end{bmatrix}
\]