WEEK 5

NONLINEAR EQUATIONS

Find a root $x_*$ of $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$f(x_*) = 0.$$ 

Example

Given

radius $- R$

area $- A$

Find $\theta$ such that

$$A(\theta) = A$$

Explicit formula for $A(\theta)$

$$A(\theta) = \frac{\theta}{2} \cdot R^2 - \frac{1}{2} R^2 \cdot \sin(\theta)$$

Thus find $\theta$ such that

$$\frac{\theta}{2} \cdot R^2 - \frac{R^2}{2} \cdot \sin(\theta) = A$$
Bisection Method (Assume $f$ is continuous)

Suppose $[a, b]$ is such that

$$f(a) \cdot f(b) < 0$$

By continuity $f$ must have a root $x_*$ on $[a, b]$.

Define $x_1 = (a + b) / 2$

1. If $f(a) \cdot f(x_1) \leq 0$
   
   $[a, x_1]$ contains a root

2. If $f(x_1) \cdot f(b) \leq 0$
   
   $[x_1, b]$ contains a root
Algorithm (Bisection)

Input: $f: \mathbb{R} \to \mathbb{R}$ continuous
$a, b \in \mathbb{R}$ such that $f(a) \cdot f(b) \leq 0$

Output: A sequence $\{x_k\}$ such that
$\lim_{k \to \infty} x_k = x^*$ satisfies $f(x^*) = 0$.

for $k = 1, 2, 3, \ldots$

$x_k = (a + b) / 2$

if $f(a) \cdot f(x_k) \leq 0$

$b = x_k$

else (otherwise $f(x_k) \cdot f(b) \leq 0$)

$a = x_k$

end

end

Example

To compute $\sqrt{2}$ find (positive) root of

$f(x) = x^2 - 2$

Choose $[a, b]$ such that $a = 0, \ b = 2$

$f(0) = -2 < 0$ and $f(2) = 2 > 0$

thus $f(0) \cdot f(2) < 0$
\[ x_1 = 1 \quad f(x_1) = -1 \quad (\text{thus search on } [0.25]) \]
\[ x_2 = 1.5 \quad f(x_2) = 0.25 \quad (\text{search on } [1, 1.5]) \]
\[ x_3 = 1.25 \quad f(x_3) = -0.4375 \quad (\text{search on } [1.25, 1.5]) \]
\[ x_4 = 1.375 \quad f(x_4) = -0.1094 \quad (\text{search on } [1.375, 1.5]) \]
\[ x_5 = 1.4375 \quad f(x_5) = 0.06640625 \quad (\text{search on } [1.375, 1.4375]) \]

Contains \( \sqrt{2} \)

Good — converges to a root provided \( f(a) \cdot f(b) \leq 0 \)

Bad — converges slow
Newton's Method (Assume $f$ has second derivatives)

Start with an initial guess $x_0$ for a root of $f$

Approximate $f(x)$ with its linear approximation (tangent line) about $x_0$

$$l(x) = f(x_0) + f'(x_0)(x-x_0)$$

Define $x_1$ as the root of $l(x)$ (assuming $f'(x_0) \neq 0$)

$$l(x_1) = f(x_0) + f'(x_0)(x_1-x_0) = 0$$

(isolate $x_1$)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
Newton update rule

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

use the tangent line at \( x_k \)

\[ l(x) = f(x_k) + f'(x_k) (x-x_k) \]

\( x_{k+1} \) is such that \( l(x_{k+1}) = 0 \).

Algorithm (Newton's Method)

Input: \( f: \mathbb{R} \rightarrow \mathbb{R} \) with second derivatives defined everywhere,
\( x_0 \in \mathbb{R} \) sufficiently close to a root \( x_\star \) of \( f \).

Output: A sequence \( \{x_k\} \) such that
\[ \lim_{k \to \infty} x_k = x_\star \text{ satisfies } f(x_\star) = 0. \]

for \( k = 0, 1, 2, \ldots \)
\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]
end
Example

Apply Newton's to compute $\sqrt{2}$, that is to compute positive root of

$$f(x) = x^2 - 2 \quad (f'(x) = 2x)$$

starting with $x_0 = 2$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{2}{4} = \frac{3}{2}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.5 - \frac{0.25}{3} = 1.4167$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.4167 - \frac{0.00694}{2.8333} = 1.414216$$

Exact value $\sqrt{2} \approx 1.414214$

rounded to 6 decimal digits

Good - fast convergence

Bad - initial guess $x_0$ must be close to a root.
If initial guess $x_0$ is not close to a root $x_*$, Newton's method may not converge.

**Example**

$$f(x) = -x^5 + x^3 + 4x, \quad x_0 = 1$$

$$f'(x) = -5x^4 + 3x^2 + 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{4}{2} = -1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1 - \frac{-4}{2} = 1$$

Thus

$$x_k = \begin{cases} 
-1 & \text{k is odd} \\
1 & \text{k is even}
\end{cases}$$

and $\lim_{k \to \infty} x_k$ does not exist.
Rate of Convergence

Let \( \{x_k\} \) be a sequence such that \( \lim_{k \to \infty} x_k = x^* \).

How does \( \frac{|x_{k+1} - x^*|}{e_{k+1}} \) compare with \( \frac{|x_k - x^*|}{e_k} \)?

Possibilities that are likely

1. \( |x_{k+1} - x^*| \) is about \( |x_k - x^*| \)
   
   **LINEAR CONVERGENCE (SLOW)**

2. \( |x_{k+1} - x^*| \) is about \( |x_k - x^*|^2 \)
   
   Thus \( |x_{k+1} - x^*| \ll |x_k - x^*| \)

   **QUADRATIC CONVERGENCE (FAST)**

3. \( |x_{k+1} - x^*| \) is about \( |x_k - x^*|^3 \)
   
   Thus \( |x_{k+1} - x^*| \ll \ll |x_k - x^*| \)

   **CUBIC CONVERGENCE (VERY FAST)**
Formally if

\( \{x_k\} \) converges to \( x_\star \) linearly if

\[
\exists c \in (0, 1) \text{ and } \exists K \in \mathbb{N} \text{ such that} \\
\Downarrow \text{real number} \\
\frac{|x_{k+1} - x_\star|}{|x_k - x_\star|} < c \quad \forall k \geq K
\]

for all \( k \) sufficiently large.

e.g.

\( \{10^{-k}\} \) converges to 0 as \( k \to \infty \)

\[
\frac{|10^{-(k+1)} - 0|}{|10^{-k} - 0|} = \frac{1}{10^c} \quad \forall k
\]

Thus convergence is linear.
\{ x_k \} converges to \( x_\ast \) quadratically if

\[ \exists C > 0 \quad \text{and} \quad \exists K \in \mathbb{N} \quad \text{such that} \]

\[ \frac{|x_{k+1} - x_\ast|}{|x_k - x_\ast|^2} \leq C \quad \forall k \geq K \]

\[ \text{e.g.} \]

\[ \{ 10^{-2^k} \} \text{ converges to 0 as } k \to \infty \]

\[ \frac{|10^{-2^{k+1}} - 0|}{|10^{-2^k} - 0|} = 10^{-2^k} \to 0 \text{ as } k \to \infty \]

Thus, convergence is faster than linear

\[ \frac{|10^{-2^{k+1}} - 0|}{|10^{-2^k} - 0|^2} = \frac{10^{-2^{k+1}}}{10^{-2^k} \cdot 10^{-2^k}} = \frac{1}{C} \quad \forall k \]

Thus, convergence is quadratic.
\( \{x_k\} \) converges to \( x_* \) cubically if

\[ \exists c > 0 \text{ and } \exists K \in \mathbb{N} \text{ such that } \]

\[ \frac{|x_{k+1} - x_*|}{|x_k - x_*|^3} \leq c \quad \forall k \geq K \]

E.g., \( \{10^{-3k^2}\} \) converges to 0 cubically.

(Verification - exercise)

Example

Suppose that the Newton sequence \( \{x_k\} \) with

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

converges to the root \( x_* = \sqrt{2} \) for \( f(x) = x^2 - 2 \).

Show that the convergence is quadratic.
Notice the recurrence
\[ x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k} \]
\[ = \frac{x_k^2 + 2}{2x_k} \cdot \]

Apply the definition of quadratic convergence
\[ \frac{|x_{k+1} - x_*|}{|x_k - x_*|^2} = \frac{|(x_k^2 + 2) - \sqrt{2}|}{|x_k - \sqrt{2}|^2} \]
\[ = \frac{|x_k^2 - 2\sqrt{2}x_k + 2|}{12x_k^2 - |x_k - \sqrt{2}|^2} \]
\[ = \frac{1}{12x_k} \quad (\text{since } \lim_{k \to \infty} x_k = \sqrt{2} > 1) \]

For all large \( k \) we have \( x_k > 1 \), thus \( \frac{1}{12x_k} \ll \frac{1}{2} \) meaning
\[ \frac{|x_{k+1} - x_*|}{|x_k - x_*|^2} \ll \frac{1}{2} \quad \forall k \text{ large enough}. \]

This indicates a quadratic convergence.
Systems of Nonlinear Equations

Given \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \), find \( x^* \) such that

\[
F(x^*) = 0 \\
\text{in } \mathbb{R}^n \\
\text{in } \mathbb{R}^n
\]

Example

Find an intersection point of the curves

- (catenary curve) \( y = \frac{1}{2} (e^{x/2} + e^{-x/2}) \)
- (ellipse) \( 9x^2 + 25y^2 = 225 \)

Let \( F: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[
F(x, y) = \begin{bmatrix}
\frac{1}{2} e^{x/2} + \frac{1}{2} e^{-x/2} - y \\
9x^2 + 25y^2 - 225
\end{bmatrix}
\]
Find \((x_*, y_*) \in \mathbb{R}^2\) such that
\[
F(x_*, y_*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Newton's method for solving a system of nonlinear equations

Jacobian matrix

\[F: \mathbb{R}^n \to \mathbb{R}^n, \quad F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_n(x) \end{bmatrix}\]

where \(F_j(x)\) is \(j\)th component of \(F(x)\)

\[
F'(x) = \begin{bmatrix}
\frac{\partial F_1(x)}{\partial x_1} & \frac{\partial F_1(x)}{\partial x_2} & \cdots & \frac{\partial F_1(x)}{\partial x_n} \\
\frac{\partial F_2(x)}{\partial x_1} & \frac{\partial F_2(x)}{\partial x_2} & \cdots & \frac{\partial F_2(x)}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n(x)}{\partial x_1} & \frac{\partial F_n(x)}{\partial x_2} & \cdots & \frac{\partial F_n(x)}{\partial x_n}
\end{bmatrix}
\]

\[
\frac{\partial F_j(x)}{\partial x_k} \quad \text{partial derivative of } F_j(x) \text{ with respect to } x_k
\]
Example

\[ F(x, y) = \begin{bmatrix} \frac{1}{2} e^{x/2} + \frac{1}{2} e^{-x/2} - y \rightarrow F_1(x, y) \\ 9x^2 + 25y^2 - 225 \rightarrow F_2(x, y) \end{bmatrix} \]

\[ F'(x, y) = \begin{bmatrix} \frac{\partial F_1(x, y)}{\partial x} & \frac{\partial F_1(x, y)}{\partial y} \\ \frac{\partial F_2(x, y)}{\partial x} & \frac{\partial F_2(x, y)}{\partial y} \end{bmatrix} \]

\[ = \begin{bmatrix} \frac{1}{4} e^{x/2} - \frac{1}{4} e^{-x/2} & -1 \\ 18x & 50y \end{bmatrix} \]

2x2 Jacobian matrix

Linear approximation about a given point \( x_k \in \mathbb{R}^n \) (generalization of tangent line at a given \( x_k \in \mathbb{R} \))

\[ (L : \mathbb{R}^n \rightarrow \mathbb{R}^n) \quad L(x) = F(x_k) + \frac{F'(x_k)}{\text{Jacobian}} (x - x_k) \]

Notes

1. \( L(x_k) = F(x_k) \)
2. \( L'(x_k) = F'(x_k) \) (Jacobian of \( L \) at \( x_k \))
Example

Linear approximation for

\[ F(x, y) = \begin{bmatrix} \frac{1}{2} e^{x/2} + \frac{1}{2} e^{-x/2} - y \\ 9x^2 + 25y^2 - 225 \end{bmatrix} \]

about \((1, 1)\).

\[ F'(1, 1) = \begin{bmatrix} \frac{1}{4} \sqrt{e} - \frac{1}{4 \sqrt{e}} & -1 \\ 18 & 50 \end{bmatrix} \]

\[ F(1, 1) = \begin{bmatrix} \frac{1}{2} \sqrt{e} + \frac{1}{2 \sqrt{e}} & -1 \\ 18 & 9 \end{bmatrix} \]

\[ L: \mathbb{R}^2 \to \mathbb{R}^2 \text{ defined by} \]

\[ L(x, y) = F(1, 1) + F'(1, 1) \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \]

\[ = \begin{bmatrix} \frac{\sqrt{e}}{2} + \frac{1}{2 \sqrt{e}} & -1 \\ 18 & 9 \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{e}}{4} - \frac{1}{4 \sqrt{e}} & -1 \\ 18 & 50 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} \]

\[ = \begin{bmatrix} \left( \frac{\sqrt{e}}{4} - \frac{1}{4 \sqrt{e}} \right)x - y + \frac{\sqrt{e}}{4} + \frac{3}{4 \sqrt{e}} \\ 18x + 50y \end{bmatrix} \]
Newton update rule

Given an estimate $x_k \in \mathbb{R}^n$ for a root; $x_{k+1}$ is the root of linear approximation $L(x)$ about $x_k$. (Assume $F'(x_k)$ is invertible)

$$L(x_{k+1}) = 0 = F(x_k) + F'(x_k)(x_{k+1} - x_k)$$

$$\Rightarrow \quad Isolate \quad x_{k+1}$$

$$x_{k+1} = x_k - \left[ F'(x_k) \right]^{-1} F(x_k)$$

Algorithm (Newton's Method for Systems)

Input : $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with well-defined second derivatives

$x_0 \in \mathbb{R}^n$ close to a root of $F$

Output : A sequence $\{x_k\}$ with $\lim_{k \to \infty} x_k = x_*$

satisfying $F(x_*) = 0.$

for $k = 0, 1, 2, \ldots$

Solve the linear system

$F'(x_k) p_k = -F(x_k)$ for $p_k$

$\quad x_{k+1} = x_k + p_k$

end
Example

Apply one iteration of Newton's method to \( F: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[
F(x, y) = \begin{bmatrix}
(x-1)^2 + (y-2)^2 + x(y+1) - 1 \\
x^2 + (y+1)^2 + (x+1)y + 2
\end{bmatrix}
\]

starting with \( x_0 = (0, 0) \).

\[
F'(x, y) = \begin{bmatrix}
2(x-1) + (y+1) & 2(y-2) + x \\
2x + y & 2(y+1) + (x+1)
\end{bmatrix}
\]

\[
F'(0, 0) = \begin{bmatrix}
-1 & -4 \\
0 & 3
\end{bmatrix}
\]

\[
F(0, 0) = \begin{bmatrix}
-1 \\
2
\end{bmatrix}
\]

Solve

\[
F'(0, 0) \ p_0 = -F(0, 0)
\]

\[
\begin{bmatrix}
-1 & -4 \\
0 & 3
\end{bmatrix} \begin{bmatrix}
p_0
\end{bmatrix} = \begin{bmatrix}
1 \\
-2
\end{bmatrix}
\]

for \( p_0 \). Indeed \( p_0 = \begin{bmatrix}
5/3 \\
-2/3
\end{bmatrix} \).

\[
x_1 = x_0 + p_0
\]

\[
= \begin{bmatrix}
0 \\
0
\end{bmatrix} + \begin{bmatrix}
5/3 \\
-2/3
\end{bmatrix} = \begin{bmatrix}
5/3 \\
-2/3
\end{bmatrix}
\]