which is usually written in the vector form

\[ |\nabla S|^2 = n^2, \]

(1.3-18)

Eikonal Equation

where \(|\nabla S|^2 = \nabla S \cdot \nabla S\). The proof of the eikonal equation from Fermat's principle is a mathematical exercise that lies beyond the scope of this book. Fermat's principle (and the ray equation) can also be shown to follow from the eikonal equation. Therefore, either the eikonal equation or Fermat's principle may be regarded as the principal postulate of ray optics.

Integrating the eikonal equation (1.3-18) along a ray trajectory between two points \(A\) and \(B\) gives

\[ S(r_B) - S(r_A) = \int_A^B |\nabla S| \, ds = \int_A^B n \, ds = \text{optical path length between } A \text{ and } B. \]

This means that the difference \(S(r_B) - S(r_A)\) represents the optical path length between \(A\) and \(B\). In the electrostatics analogy, the optical path length plays the role of the potential difference.

To determine the ray trajectories in an inhomogeneous medium of refractive index \(n(r)\), we can either solve the ray equation (1.3-2), as we have done earlier, or solve the eikonal equation for \(S(r)\), from which we calculate the gradient \(\nabla S\).

If the medium is homogeneous, i.e., \(n(r)\) is constant, the magnitude of \(\nabla S\) is constant, so that the wavefront normals (rays) must be straight lines. The surfaces \(S(r) = \text{constant}\) may be parallel planes or concentric spheres, as illustrated in Fig. 1.3-10.

### 1.4 MATRIX OPTICS

Matrix optics is a technique for tracing paraxial rays. The rays are assumed to travel only within a single plane, so that the formalism is applicable to systems with planar geometry and to meridional rays in circularly symmetric systems.

A ray is described by its position and its angle with respect to the optical axis. These variables are altered as the ray travels through the system. In the paraxial approximation, the position and angle at the input and output planes of an optical system are

---

Rays

medium.

\[ (1.3-18) \]
Elkonin Equation

It's principle is a principle (andtion). Therefore, as the principal
seen two points

en A and B.

cal path length th plays the role
refractive index lier, or solve the
altitude of VS is z. The surfaces illustrated in Fig.

assumed to travel xems with planar
optical axis. These axial approxima-
ptical system are

related by two linear algebraic equations. As a result, the optical system is described by a \( 2 \times 2 \) matrix called the ray-transfer matrix.

The convenience of using matrix methods lies in the fact that the ray-transfer matrix of a cascade of optical components (or systems) is a product of the ray-transfer matrices of the individual components (or systems). Matrix optics therefore provides a formal mechanism for describing complex optical systems in the paraxial approximation.

A. The Ray-Transfer Matrix

Consider a circularly symmetric optical system formed by a succession of refracting and reflecting surfaces all centered about the same axis (optical axis). The \( z \) axis lies along the optical axis and points in the general direction in which the rays travel. Consider rays in a plane containing the optical axis, say the \( y-z \) plane. We proceed to trace a ray as it travels through the system, i.e., as it crosses the transverse planes at different axial distances. A ray crossing the transverse plane at \( z \) is completely characterized by the coordinate \( y \) of its crossing point and the angle \( \theta \) (Fig. 1.4-1).

An optical system is a set of optical components placed between two transverse planes at \( z_1 \) and \( z_2 \), referred to as the input and output planes, respectively. The system is characterized completely by its effect on an incoming ray of arbitrary position and direction \( (y_1, \theta_1) \). It steers the ray so that it has new position and direction \( (y_2, \theta_2) \) at the output plane (Fig. 1.4-2).

\[ \text{Figure 1.4-1} \quad \text{A ray is characterized by its coordinate } y \text{ and its angle } \theta. \]

\[ \text{Figure 1.4-2} \quad \text{A ray enters an optical system at position } y_1 \text{ and angle } \theta_1 \text{ and leaves at position } y_2 \text{ and angle } \theta_2. \]
In the paraxial approximation, when all angles are sufficiently small so that \( \sin \theta \approx \theta \), the relation between \((y_2, \theta_2)\) and \((y_1, \theta_1)\) is linear and can generally be written in the form

\[
\begin{align*}
y_2 &= Ay_1 + B\theta_1, \quad (1.4-1) \\
\theta_2 &= Cy_1 + D\theta_1, \quad (1.4-2)
\end{align*}
\]

where \(A, B, C\), and \(D\) are real numbers. Equations (1.4-1) and (1.4-2) may be conveniently written in matrix form as

\[
\begin{bmatrix}
y_2 \\
\theta_2
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
y_1 \\
\theta_1
\end{bmatrix}.
\]

The matrix \(M\), whose elements are \(A, B, C, D\), characterizes the optical system completely since it permits \((y_2, \theta_2)\) to be determined for any \((y_1, \theta_1)\). It is known as the ray-transfer matrix.

**Exercise 1.4-1**

**Special Forms of the Ray-Transfer Matrix.** Consider the following situations in which one of the four elements of the ray-transfer matrix vanishes:

(a) Show that if \(A = 0\), all rays that enter the system at the same angle leave at the same position, so that parallel rays in the input are focused to a single point at the output.

(b) What are the special features of each of the systems for which \(B = 0\), \(C = 0\), or \(D = 0\)?

**B. Matrices of Simple Optical Components**

**Free-Space Propagation**

Since rays travel in free space along straight lines, a ray traversing a distance \(d\) is altered in accordance with \(y_2 = y_1 + \theta_1 d\) and \(\theta_2 = \theta_1\). The ray-transfer matrix is therefore

\[
M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}. \quad (1.4-3)
\]

**Refraction at a Planar Boundary**

At a planar boundary between two media of refractive indices \(n_1\) and \(n_2\), the ray angle changes in accordance with Snell’s law \(n_1 \sin \theta_1 = n_2 \sin \theta_2\). In the paraxial approximation, \(n_1 \theta_1 = n_2 \theta_2\). The position of the ray is not altered, \(y_2 = y_1\). The ray-transfer
so that \( \sin \theta = \theta \),
be written in the

\[
\begin{align*}
(1.4-1) \\
(1.4-2)
\end{align*}
\]

(1.4-2) may be optical system is known as the

tions in which
we at the same at the output.
0, \( C = 0 \), or

distance \( d \) is transfer matrix is

\[
\begin{align*}
(1.4-3)
\end{align*}
\]

, the ray angle approximate ray-transfer

matrix is

\[
M = \begin{bmatrix}
1 & 0 \\
0 & \frac{n_1}{n_2}
\end{bmatrix}.
\]

(1.4-4)

**Refraction at a Spherical Boundary**
The relation between \( \theta_1 \) and \( \theta_2 \) for paraxial rays refracted at a spherical boundary between two media is provided in (1.2-8). The ray height is not altered, \( y_2 = y_1 \). The ray-transfer matrix is

\[
M = \begin{bmatrix}
\frac{1}{n_2 - n_1} & 0 \\
-\frac{n_1}{n_2 R} & \frac{n_1}{n_2}
\end{bmatrix}.
\]

(1.4-5)

**Transmission Through a Thin Lens**
The relation between \( \theta_1 \) and \( \theta_2 \) for paraxial rays transmitted through a thin lens of focal length \( f \) is given in (1.2-11). Since the height remains unchanged \( y_2 = y_1 \),

\[
M = \begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix}.
\]

(1.4-6)

**Reflection from a Planar Mirror**
Upon reflection from a planar mirror, the ray position is not altered, \( y_2 = y_1 \). Adopting the convention that the \( z \) axis points in the general direction of travel of the rays, i.e., toward the mirror for the incident rays and away from it for the reflected rays, we conclude that \( \theta_2 = \theta_1 \). The ray-transfer matrix is therefore the identity matrix

\[
M = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

(1.4-7)

**Reflection from a Spherical Mirror**
Using (1.2-1), and the convention that the \( z \) axis follows the general direction of the rays as they reflect from mirrors, we similarly obtain

\[
M = \begin{bmatrix}
1 & 0 \\
\frac{-1}{R} & 1
\end{bmatrix}.
\]

(1.4-8)

Convex, \( R > 0 \); concave, \( R < 0 \)
Note the similarity between the ray-transfer matrices of a spherical mirror (1.4-8) and a thin lens (1.4-6). A mirror with radius of curvature $R$ bends rays in a manner that is identical to that of a thin lens with focal length $f = -R/2$.

C. Matrices of Cascaded Optical Components

A cascade of optical components whose ray-transfer matrices are $M_1, M_2, \ldots, M_N$ is equivalent to a single optical component of ray-transfer matrix

$$M = M_N \cdots M_2 M_1.$$  \hspace{1cm} (1.4-9)

Note the order of matrix multiplication: The matrix of the system that is crossed by the rays first is placed to the right, so that it operates on the column matrix of the incident ray first.

**EXERCISE 1.4-2**

_A Set of Parallel Transparent Plates._ Consider a set of $N$ parallel planar transparent plates of refractive indices $n_1, n_2, \ldots, n_N$ and thicknesses $d_1, d_2, \ldots, d_N$, placed in air ($n = 1$) normal to the $z$ axis. Show that the ray-transfer matrix is

$$M = \begin{bmatrix} 1 & \sum_{i=1}^{N} \frac{d_i}{n_i} \\ 0 & 1 \end{bmatrix}. \hspace{1cm} (1.4-10)$$

Note that the order of placing the plates does not affect the overall ray-transfer matrix. What is the ray-transfer matrix of an inhomogeneous transparent plate of thickness $d_0$ and refractive index $n(z)$?

**EXERCISE 1.4-3**

_A Gap Followed by a Thin Lens._ Show that the ray-transfer matrix of a distance $d'$ of free space followed by a lens of focal length $f$ is

$$M = \begin{bmatrix} 1 & \frac{d'}{f} \\ -\frac{f}{d'} & 1 - \frac{f^2}{d'^2} \end{bmatrix}. \hspace{1cm} (1.4-11)$$

**EXERCISE 1.4-4**

_Imaging with a Thin Lens._ Derive an expression for the ray-transfer matrix of a system comprised of free space/thin lens/free space, as shown in Fig. 1.4-3. Show that if the
Imaging condition \(1/d_1 + 1/d_2 = 1/f\) is satisfied, all rays originating from a single point in the input plane reach the output plane at the single point \(y_2\), regardless of their angles. Also show that if \(d_2 = f\), all parallel incident rays are focused by the lens onto a single point in the output plane.

![Figure 1.4-3 Single-lens imaging system.](image)

**EXERCISE 1.4-5**

**Imaging with a Thick Lens.** Consider a glass lens of refractive index \(n\), thickness \(d\), and two spherical surfaces of equal radii \(R\) (Fig. 1.4-4). Determine the ray-transfer matrix of the system between the two planes at distances \(d_1\) and \(d_2\) from the vertices of the lens. The lens is placed in air (refractive index = 1). Show that the system is an imaging system (i.e., the input and output planes are conjugate) if

\[
\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f} \quad \text{or} \quad s_1s_2 = f^2, \tag{1.4-12}
\]

where

\[
\begin{align*}
z_1 &= d_1 + h, & s_1 &= z_1 - f, \\
z_2 &= d_2 + h, & s_2 &= z_2 - f
\end{align*} \tag{1.4-13}
\]

and

\[
h = \frac{(n - 1)d}{nR}, \tag{1.4-14}
\]

\[
\frac{1}{f} = \frac{(n - 1)}{R} \left[ 2 - \frac{n - 1}{n} \right]. \tag{1.4-15}
\]

The points \(F_1\) and \(F_2\) are known as the front and back focal points, respectively. The points \(P_1\) and \(P_2\) are known as the first and second principal points, respectively. Show the importance of these points by tracing the trajectories of rays that are incident parallel to the optical axis.

![Figure 1.4-4 Imaging with a thick lens. \(P_1\) and \(P_2\) are the principal points and \(F_1\) and \(F_2\) are the focal points.](image)
D. Periodic Optical Systems

A periodic optical system is a cascade of identical unit systems. An example is a sequence of equally spaced identical relay lenses used to guide light, as shown in Fig. 1.2-16(a). Another example is the reflection of light between two parallel mirrors forming an optical resonator (see Chap. 9); in that case, the ray traverses the same unit system (a round trip of reflections) repeatedly. A homogeneous medium, such as a glass fiber, may be considered as a periodic system if it is divided into contiguous identical segments of equal length. A general theory of ray propagation in periodic optical systems will now be formulated using matrix methods.

**Difference Equation for the Ray Position**

A periodic system is composed of a cascade of identical unit systems (stages), each with a ray-transfer matrix \( (A, B, C, D) \), as shown in Fig. 1.4-5. A ray enters the system with initial position \( y_0 \) and slope \( \theta_0 \). To determine the position and slope \( (y_m, \theta_m) \) of the ray at the exit of the \( m \)th stage, we apply the \( ABCD \) matrix \( m \) times,

\[
\begin{bmatrix}
  y_m \\
  \theta_m
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^m \begin{bmatrix} y_0 \\
  \theta_0
\end{bmatrix}.
\]

We can also apply the relations

\[
y_{m+1} = Ay_m + B\theta_m, \quad \theta_{m+1} = Cy_m + D\theta_m
\]

iteratively to determine \( (y_1, \theta_1) \) from \( (y_0, \theta_0) \), then \( (y_2, \theta_2) \) from \( (y_1, \theta_1) \), and so on, using a computer.

It is of interest to derive equations that govern the dynamics of the position \( y_m \), \( m = 0, 1, \ldots \), irrespective of the angle \( \theta_m \). This is achieved by eliminating \( \theta_m \) from (1.4-17) and (1.4-18). From (1.4-17)

\[
\theta_m = \frac{y_{m+1} - Ay_m}{B}.
\]

Replacing \( m \) with \( m + 1 \) in (1.4-19) yields

\[
\theta_{m+1} = \frac{y_{m+2} - Ay_{m+1}}{B}.
\]

Substituting (1.4-19) and (1.4-20) into (1.4-18) gives

\[
y_{m+2} = 2by_{m+1} - F^2y_m,
\]

Recurrence Relation for Ray Position

**Figure 1.4-5** A cascade of identical optical components.
where

\[ b = \frac{A + D}{2} \]  

\[ F^2 = AD - BC = \det[M], \]  

and \( \det[M] \) is the determinant of \( M \).

Equation (1.4-21) is a linear difference equation governing the ray position \( y_m \). It can be solved iteratively on a computer by computing \( y_2 \) from \( y_0 \) and \( y_1 \), then \( y_3 \) from \( y_1 \) and \( y_2 \), and so on. \( y_1 \) may be computed from \( y_0 \) and \( \theta_0 \) by use of (1.4-17) with \( m = 0 \).

It is useful, however, to derive an explicit expression for \( y_m \) by solving the difference equation (1.4-21). As in linear differential equations, a solution satisfying a linear difference equation and the initial conditions is a unique solution. It is therefore appropriate to make a judicious guess for the solution of (1.4-21). We use a trial solution of the geometric form

\[ y_m = y_0 h^m, \]  

(1.4-24)

where \( h \) is a constant. Substituting (1.4-24) into (1.4-21) immediately shows that the trial solution is suitable provided that \( h \) satisfies the quadratic algebraic equation

\[ h^2 - 2bh + F^2 = 0, \]  

(1.4-25)

from which

\[ h = b \pm (F^2 - b^2)^{1/2}. \]  

(1.4-26)

The results can be presented in a more compact form by defining the variable

\[ \varphi = \cos^{-1} \frac{b}{F}, \]  

(1.4-27)

so that \( b = F \cos \varphi \), \((F^2 - b^2)^{1/2} = F \sin \varphi \), and therefore \( h = F(\cos \varphi \pm j \sin \varphi) = F \exp(\pm j \varphi) \), whereupon (1.4-24) becomes \( y_m = y_0 F^m \exp(\pm j m \varphi) \).

A general solution may be constructed from the two solutions with positive and negative signs by forming their linear combination. The sum of the two exponential functions can always be written as a harmonic (circular) function, so that

\[ y_m = y_{\text{max}} F^m \sin(m \varphi + \varphi_0), \]  

(1.4-28)

where \( y_{\text{max}} \) and \( \varphi_0 \) are constants to be determined from the initial conditions \( y_0 \) and \( y_1 \). In particular, \( y_{\text{max}} = y_0 / \sin \varphi_0 \).

The parameter \( F \) is related to the determinant of the ray-transfer matrix of the unit system by \( F = \det^{1/2}[M] \). It can be shown that regardless of the unit system, \( \det[M] = n_1/n_2 \), where \( n_1 \) and \( n_2 \) are the refractive indices of the initial and final sections of the unit system. This general result is easily verified for the ray-transfer matrices of all the optical components considered in this section. Since the determinant of a product of two matrices is the product of their determinants, it follows that the relation \( \det[M] = n_1/n_2 \) is applicable to any cascade of these optical components. For example, if \( \det[M_1] = n_1/n_2 \) and \( \det[M_2] = n_2/n_3 \), then \( \det[M_2M_1] = (n_2/n_3)(n_1/n_2) = n_1/n_3 \).
Ray Optics

In most applications \( n_1 = n_2 \), so that \( \det(M) = 1 \) and \( F = 1 \), in which case the solution for the ray position is

\[
y_m = y_{\text{max}} \sin \left( m \varphi + \varphi_0 \right) .
\]

(1.4-29)

Ray Position in a Periodic System

We shall assume henceforth that \( F = 1 \).

**Condition for a Harmonic Trajectory**

For \( y_m \) to be a harmonic (instead of hyperbolic) function, \( \varphi = \cos^{-1} b \) must be real. This requires that

\[
|b| \leq 1 \quad \text{or} \quad \frac{|A + D|}{2} \leq 1.
\]

(1.4-30)

Condition for a Stable Solution

If, instead, \( |b| > 1 \), \( \varphi \) is then imaginary and the solution is a hyperbolic function (cosh or sinh), which increases without bound, as illustrated in Fig. 1.4-6(a). A harmonic solution ensures that \( y_m \) is bounded for all \( m \), with a maximum value of \( y_{\text{max}} \). The bound \( |b| \leq 1 \) therefore provides a condition of stability (boundedness) of the ray trajectory.

The ray angle corresponding to (1.4-29) is also a harmonic function \( \theta_m = \theta_{\text{max}} \sin (m \varphi + \varphi_0) \), where \( \theta_{\text{max}} \) and \( \varphi_0 \) are constants. This can be shown by use of (1.4-19) and trigonometric identities. The maximum angle \( \theta_{\text{max}} \) must be sufficiently small so that the paraxial approximation, which underlies this analysis, is applicable.

**Condition for a Periodic Trajectory**

The harmonic function (1.4-29) is periodic in \( m \) if it is possible to find an integer \( s \) such that \( y_{m+s} = y_m \) for all \( m \). The smallest such integer is the period. The ray then

![Figure 1.4-6](image-url)

Figure 1.4-6 Examples of trajectories in periodic optical systems: (a) unstable trajectory \( b > 1 \); (b) stable and periodic trajectory \( \varphi = \frac{6\pi}{11}; \text{ period } = 11 \text{ stages} \); (c) stable but nonperiodic trajectory \( \varphi = 1.2 \).
retraces its path after \( s \) stages. This condition is satisfied if \( s \varphi = 2 \pi q \), where \( q \) is an integer. Thus the necessary and sufficient condition for a periodic trajectory is that \( \varphi/2\pi \) is a rational number \( q/s \). If \( \varphi = 6\pi/11 \), for example, then \( \varphi/2\pi = \frac{3}{11} \) and the trajectory is periodic with period \( s = 11 \) stages. This case is illustrated in Fig. 1.4.6(b).

**Summary**

A paraxial ray traveling through a cascade of identical unit optical systems, each with a ray-transfer matrix with elements \((A, B, C, D)\) such that \(AD=BC=1\), follows a harmonic (and therefore bounded) trajectory if the condition \((A+D)/2 \leq 1\), called the stability condition, is satisfied. The position at the \( m \)th stage is then \( y_m = y_{\text{max}} \sin(m\varphi + \varphi_0) \), \( m = 0, 1, 2, \ldots \), where \( \varphi = \cos^{-1}(A+D)/2 \). The constants \( y_{\text{max}} \) and \( \varphi_0 \) are determined from the initial positions \( y_0 \) and \( y_1 = Ay_0 + B\varphi_0 \), where \( \varphi_0 \) is the initial ray inclination. The ray angles are related to the positions by \( \theta_m = (y_{m} + A\varphi_{m})/B \) and follow a harmonic function \( \theta_m = \theta_{\text{max}} \sin(m\varphi + \varphi_1) \). For the paraxial approximation to be valid, \( \theta_{\text{max}} \ll 1 \). The ray trajectory is periodic with period \( s \) if \( \varphi/2\pi \) is a rational number \( q/s \).

**Example 1.4.1. A Sequence of Equally Spaced Identical Lenses.** A set of identical lenses of focal length \( f \) separated by distance \( d \), as shown in Fig. 1.4.7, may be used to relay light between two locations. The unit system, a distance \( d \) of free space followed by a lens, has a ray-transfer matrix given by (1.4-11): \( A = 1, B = d, C = -1/f, D = 1 - d/2f \).

The parameter \( b = (A+D)/2 = 1 - d/2f \) and the determinant is unity. The condition for a stable ray trajectory, \( |b| \leq 1 \) or \( -1 \leq b \leq 1 \), is therefore

\[
0 \leq d \leq 4f,
\]

so that the spacing between the lenses must be smaller than four times the focal length. Under this condition the positions of paraxial rays obey the harmonic function

\[
y_m = y_{\text{max}} \sin(m\varphi + \varphi_0), \quad \varphi = \cos^{-1}\left(1 - \frac{d}{2f}\right).
\]

When \( d = 2f \), \( \varphi = \pi/2 \) and \( \varphi/2\pi = \frac{1}{4} \), so that the trajectory of an arbitrary ray is

![Figure 1.4.7](image_url)
periodic with period equal to four stages. When $d = f_1$, $\varphi = \pi f / 3$ and $\varphi / 2\pi = 1 / 6$, so that the ray trajectory is periodic and retraces itself each six stages. These cases are illustrated in Fig. 1.4-8.

\[ \text{EXERCISE 1.4-6} \]

A Periodic Set of Pairs of Different Lenses. Examine the trajectories of paraxial rays through a periodic system composed of a set of lenses with alternating focal lengths $f_1$ and $f_2$ as shown in Fig. 1.4-9. Show that the ray trajectory is bounded (stable) if

\[ 0 \leq \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right) \leq 1. \]  

\[ (1.4-33) \]

\[ \text{EXERCISE 1.4-7} \]

An Optical Resonator. Paraxial rays are reflected repeatedly between two spherical mirrors of radii $R_1$ and $R_2$ separated by a distance $d$ (Fig. 1.4-10). Regarding this as a periodic system whose unit system is a single round trip between the mirrors, determine the condition of stability of the ray trajectory. Optical resonators will be studied in detail in Chap. 9.
\(d = 2f\)

- \(\frac{1}{2}\), so that

- illustrated

General

RAY OPTICS


Geometrical Optics


Optical System Design


Matrix Optics


Popular and Historical

PROBLEMS

1.2-1 Transmission through Planar Plates. (a) Use Snell's law to show that a ray entering a planar plate of width $d$ and refractive index $n_1$ (placed in air; $n = 1$) emerges parallel to its initial direction. The ray need not be paraxial. Derive an expression for the lateral displacement of the ray as a function of the angle of incidence $\theta$. Explain your results in terms of Fermat's principle.

(b) If the plate is, instead, made of a stack of $N$ parallel layers of thicknesses $\alpha_1, \alpha_2, \ldots, \alpha_N$ and refractive indices $n_1, n_2, \ldots, n_N$, show that the transmitted ray is parallel to the incident ray. If $\theta_m$ is the angle of the ray in the $m$th layer, show that $n_m \sin \theta_m = \sin \theta_m$, $m = 1, 2, \ldots$.

1.2-2 Lens in Water. Determine the focal length $f$ of a biconvex lens with radii 20 cm and 30 cm and refractive index $n = 1.5$. What is the focal length when the lens is immersed in water ($\eta = \frac{4}{3}$)?

1.2-3 Numerical Aperture of a Claddless Fiber. Determine the numerical aperture and the acceptance angle of an optical fiber if the refractive index of the core is $n_1 = 1.46$ and the cladding is stripped out (replaced with air $n_2 = 1$).

1.2-4 Fiber Coupling Spheres. Tiny glass balls are often used as lenses to couple light into and out of optical fibers. The fiber end is located at a distance $f$ from the sphere. For a sphere of radius $a = 1$ mm and refractive index $n = 1.8$, determine $f$ such that a ray parallel to the optical axis at a distance $y = 0.7$ mm is focused onto the fiber, as illustrated in Fig. P1.2-4.

![Figure P1.2-4](image)

**Figure P1.2-4** Focusing light into an optical fiber with a spherical glass ball.

1.2-5 Extraction of Light from a High-Refractive-Index Medium. Assume that light is generated isotropically in all directions inside a material of refractive index $n = 3.7$ cut in the shape of a parallelepiped and placed in air ($n = 1$) (see Exercise 1.2-6). (a) If a reflective material acting as a perfect mirror is coated on all sides except the front side, determine the percentage of light that may be extracted from the front side.

(b) If another transparent material of refractive index $n = 1.4$ is placed on the front side, would that help extract some of the trapped light?

1.3-1 Axially Graded Plate. A plate of thickness $d$ is oriented normal to the $z$-axis. The refractive index $n(z)$ is graded in the $z$ direction. Show that a ray entering the plate from air at an incidence angle $\theta_i$ in the $y-z$ plane makes an angle $\theta(z)$ at position $z$ in the medium given by $n(z) \sin \theta(z) = \sin \theta_i$. Show that the ray emerges into air.
parallel to the original incident ray. *Hint:* You may use the results of Problem 1.2.1. Show that the ray position \( y(x) \) inside the plate obeys the differential equation 
\[
\left( \frac{dy}{dz} \right)^2 = \left( \frac{n^2}{\sin^2 \theta} - 1 \right)^{-1}.
\]

1.3-2 Ray Trajectories in GRIN Fibers. Consider a graded-index optical fiber with cylindrical symmetry about the \( z \) axis and refractive index \( n(\rho) = (x^2 + y^2)^{1/2} \). Let \((\rho, \phi, z)\) be the position vector in a cylindrical coordinate system. Rewrite the paraxial ray equations, (1.3-3), in a cylindrical system and derive differential equations for \( \rho \) and \( \phi \) as functions of \( z \).

1.4-1 Ray-Transfer Matrix of a Lens System. Determine the ray-transfer matrix for an optical system made of a thin convex lens of focal length \( f \) and a thin concave lens of focal length \(-f\) separated by a distance \( f \). Discuss the imaging properties of this composite lens.

1.4-2 Ray-Transfer Matrix of a GRIN Plate. Determine the ray-transfer matrix of a SELFOC plate [i.e., a graded-index material with parabolic refractive index \( n(y) = n_0(1 - \frac{1}{2}(y^2)) \)] of width \( d \).

1.4-3 The GRIN Plate as a Periodic System. Consider the trajectories of paraxial rays inside a SELFOC plate normal to the \( z \) axis. This system may be regarded as a periodic system made of a sequence of identical contiguous plates of thickness \( d \) each. Using the result of Problem 1.4-2, determine the stability condition of the ray trajectory. Is this condition dependent on the choice of \( d \)?

1.4-4 4 \( \times \) 4 Ray-Transfer Matrix for Skewed Rays. Matrix methods may be generalized to describe skewed paraxial rays in circularly symmetric systems, and to astigmatic (non-circularly symmetric) systems. A ray crossing the plane \( z = 0 \) is generally characterized by four variables—the coordinates \((x, y)\) of its position in the plane, and the angles \( (\theta_x, \theta_y) \) that its projections in the \( x-z \) and \( y-z \) planes make with the \( z \) axis. The emerging ray is also characterized by four variables linearly related to the initial four variables. The optical system may then be characterized completely, within the paraxial approximation, by a 4 \( \times \) 4 matrix.

(a) Determine the 4 \( \times \) 4 ray-transfer matrix of a distance \( d \) in free space.

(b) Determine the 4 \( \times \) 4 ray-transfer matrix of a thin cylindrical lens with focal length \( f \) oriented in the \( y \) direction (Fig. P1.4-4). The cylindrical lens has focal length \( f \) for rays in the \( y-z \) plane, and no focusing power for rays in the \( x-z \) plane.

![Figure P1.4-4 Cylindrical lens.](image)