1. (70pts.) A government offers a procurement contract to buy $q \geq 0$ units of a good from a firm, in exchange for a payment of $t \geq 0$. The firm’s constant unit cost is private information and is either high ($c_h$) or low ($c_l$) (with $c_h > c_l > 0$). Government’s prior belief about the firm’s cost is given by $\operatorname{prob}(c_l) = p \in (0, 1)$. If the firm with cost $c_i, i = l, h$, accepts the contract $(t_i, q_i)$, its profit is $t_i - c_i q_i$, whereas if it rejects it receives zero profit. Government’s payoff is given by $\ln q - t$.

(a) (20pts.) Assume that the government observes the firm’s type and find the first best contract.

Solution

The government’s problem is

$$\max_{(t_l, q_l), (t_h, q_h)} p(\ln q_l - t_l) + (1 - p)(\ln q_h - t_h)$$

subject to

$$t_l - c_l q_l \geq 0$$
$$t_h - c_h q_h \geq 0$$
$$t_l, t_h, q_l, q_h \geq 0$$

At the solution the IR constraints must hold as equality, i.e.,

$$t_l = c_l q_l \quad \text{and} \quad t_h = c_h q_h$$

Substituting into the objective function, the problem becomes

$$\max_{q_l, q_h} p(\ln q_l - c_l q_l) + (1 - p)(\ln q_h - c_h q_h)$$

$$q_l, q_h \geq 0$$

At the solution we must have $q_l > 0$ and $q_h > 0$ since

$$\lim_{q_i \to 0} \ln q_i - c_i q_i = -\infty, \quad i = l, h$$

Therefore, the following first order conditions must hold

$$\frac{1}{q_l} = c_l \quad \text{and} \quad \frac{1}{q_h} = c_h$$

We can solve these as

$$q_l = \frac{1}{c_l} \quad \text{and} \quad q_h = \frac{1}{c_h}$$

Therefore, $t_l = t_h = 1$

(b) (30pts.) Now assume that the government cannot observe the firm’s type and find the optimal contract. Compare it with the first best.

Solution

The government’s problem is given by

$$\max_{(t_l, q_l), (t_h, q_h)} p(\ln q_l - t_l) + (1 - p)(\ln q_h - t_h)$$

subject to

$$t_l - c_l q_l \geq 0$$
$$t_h - c_h q_h \geq 0$$
$$t_l - c_l q_l \geq t_h - c_h q_h$$
$$t_h - c_h q_h \geq t_l - c_l q_l$$
$$t_l, t_h, q_l, q_h \geq 0$$

The following must hold at the solution:
i. Individual rationality constraint of the high cost holds with equality, i.e., \( t_h - c_h q_h = 0 \).

**Proof.** Suppose, for contradiction, that \( t_h - c_h q_h > 0 \). Then, the incentive compatibility (IC) constraint of the low cost and \( c_l < c_h \) imply that
\[
t_l - c_l q_l \geq t_h - c_h q_h > t_h - c_h q_h > 0
\]
This cannot be a solution since the government can decrease \( t_l \) and \( t_h \) by the same small amount without violating any of the constraints and increase its payoff.

ii. Individual rationality constraint of the low cost does not bind at the solution.

**Proof.** IC constraint of the low cost, \( c_l < c_h \), and the IR constraint of the high cost imply that
\[
t_l - c_l q_l \geq t_h - c_l q_h > t_h - c_h q_h > 0
\]
Therefore, the government can decrease \( t_l \) by some small amount without violating any of the constraints and increase its payoff.

iii. Incentive compatibility constraint of the low cost holds with equality, i.e., \( t_l - c_l q_l = t_h - c_h q_h \).

**Proof.** Suppose, for contradiction, that \( t_l - c_l q_l > t_h - c_h q_h \). Then,
\[
t_l - c_l q_l > t_h - c_l q_h > t_h - c_h q_h > 0
\]
Therefore, the government can decrease \( t_l \) by some small amount without violating any of the constraints and increase its payoff.

iv. \( q_l \geq q_h \).

**Proof.** The two IC constraints can be written as
\[
c_l (q_l - q_h) \leq t_l - t_h \leq c_h (q_l - q_h)
\]
Since \( c_l < c_h \), we must have \( q_l \geq q_h \).

v. Incentive compatibility constraint of the high cost can be ignored.

**Proof.** Since the IC constraint for the low cost holds with equality and \( q_l \geq q_h \) we have
\[
t_l - t_h = c_l (q_l - q_h) \leq c_h (q_l - q_h)
\]
which implies
\[
t_h - c_h q_h \geq t_l - c_h q_l
\]
Therefore, the government’s problem is equivalent to the following “relaxed” problem

\[
\max_{(t_l, q_l), (t_h, q_h)} \quad p(\ln q_l - t_l) + (1 - p)(\ln q_h - t_h)
\]
subject to
\[
\begin{align*}
t_h - c_h q_h &= 0, \\
t_l - c_l q_l &= t_h - c_h q_h, \\
q_l &\geq q_h, \\
q_l, q_h &\geq 0
\end{align*}
\]
From the equality constraints we have
\[
\begin{align*}
t_h &= c_h q_h, \\
t_l &= c_l q_l + (c_h - c_l) q_h
\end{align*}
\]
Substituting into the objective function we obtain

\[
\max_{q_l, q_h} \quad p(\ln q_l - c_l q_l - (c_h - c_l) q_h) + (1 - p)(\ln q_h - c_h q_h)
\]
subject to
\[
\begin{align*}
q_l &\geq q_h, \\
q_l, q_h &\geq 0
\end{align*}
\]
Let us first ignore the monotonicity condition. At the solution we must have \( q_l > 0 \) and \( q_h > 0 \), and hence, the following first order conditions must hold:

\[
p \left( \frac{1}{q_l} - c_l \right) = 0
\]

\[
-p(c_h - c_l) + (1 - p) \left( \frac{1}{q_h} - c_h \right) = 0
\]

These can be solved as

\[
q_l = \frac{1}{c_l}
\]

\[
q_h = \frac{1}{c_h + \frac{p}{1-p}(c_h - c_l)}
\]

Since, \( c_l > c_h \), we have \( q_l > q_h \), i.e., the monotonicity condition is satisfied. Therefore, this is indeed the solution. Transfers can be calculated using

\[
t_h = c_h q_h
\]

\[
t_l = c_l q_l + (c_h - c_l) q_h
\]

The low cost type produces the efficient (first-best) quantity and receives \( (c_h - c_l) q_h \) as information rent, whereas the high cost produces less than the efficient amount and receives zero rent. High cost type’s quantity is distorted to reduce the information rent of the low cost. The amount of distortion increases in the cost difference, \( c_h - c_l \), and in the fraction of low cost types, \( p \).

(c) (20pts.) Assume that the set of possible types is \([1, 2]\) and the government believes that the cost of the firm, \( \theta \), is distributed uniformly over this interval, i.e., \( F(\theta) = \theta - 1, 1 \leq \theta \leq 2 \). Find the optimal contract for this case.

**Solution**

Optimal contracting problem is given by

\[
\max_{(q,t)} \int_1^2 (\ln q(\theta) - t(\theta))d\theta
\]

s.t.

\[
t(\theta) - \theta q(\theta) \geq 0, \quad \forall \theta
\]

\[
t(\theta) - \theta q(\theta) \geq t(\theta') - \theta q(\theta'), \quad \forall \theta, \theta'
\]

Let

\[
U(\theta) = t(\theta) - \theta q(\theta).
\]

We first show that incentive compatibility constraints are satisfied if and only if

i. \( q \) is decreasing

ii. \( U(\theta) = U(1) - \int_1^\theta q(\tau)d\tau \), for any \( \theta \)

**Proof.** Using the definition of \( U(\theta) \) we can write the incentive compatibility constraints for any \( \theta > \theta' \) as follows:

\[
U(\theta) \geq U(\theta') + q(\theta')(\theta' - \theta)
\]

\[
U(\theta') \geq U(\theta) + q(\theta)(\theta - \theta')
\]

Equivalently,

\[
q(\theta')(\theta' - \theta) \leq U(\theta) - U(\theta') \leq q(\theta)(\theta' - \theta)
\]

Therefore, \( \theta' > \theta \) implies that \( q(\theta) \geq q(\theta') \). Dividing by \( \theta' - \theta \) and taking the limit as \( \theta' \to \theta \), we get

\[
\frac{dU(\theta)}{d\theta} = -q(\theta)
\]

Therefore,

\[
U(\theta) = U(1) - \int_1^\theta q(\tau)d\tau
\]
Suppose now that $q$ is decreasing and $U(\theta) = U(1) - \int_1^{\theta} q(\tau)d\tau$. Let $\theta' > \theta$. We have

$$U(\theta) - U(\theta') = \int_1^{\theta'} q(\tau)d\tau - \int_1^{\theta'} q(\tau)d\tau = \int_0^{\theta'} q(\tau)d\tau \geq q(\theta')(\theta' - \theta)$$

and hence the IC constraints are satisfied.

Note that IR constraint of type $\theta = 2$, i.e., $U(2) \geq 0$ implies that

$$U(1) - \int_1^{2} q(\tau)d\tau = U(2) \geq 0.$$  

Therefore,

$$U(\theta) = U(1) - \int_1^{\theta} q(\tau)d\tau \geq \int_1^{2} q(\tau)d\tau - \int_1^{\theta} q(\tau)d\tau = \int_1^{2} q(\tau)d\tau \geq 0.$$  

In other words, IR constraint of type $\theta = 2$ implies all the other IR constraints. Therefore, the problem is equivalent to

$$\max_{(q,t)} \int_1^{2} \ln q(\theta) - t(\theta) d\theta$$  

s.t. $U(2) \geq 0$  

$$U(\theta) = U(1) - \int_1^{\theta} q(\tau)d\tau$$  

$q$ is decreasing

Using the definition of $U(\theta)$ and substituting we can write the objective function as

$$\int_1^{2} \left[ \ln q(\theta) - U(1) - \theta q(\theta) + \int_1^{\theta} q(\tau)d\tau \right] d\theta.$$  

Substituting for $U(1)$ using

$$U(1) = U(2) + \int_1^{2} q(\tau)d\tau$$  

The objective function can be written as

$$\int_1^{2} \left[ \ln q(\theta) - \theta q(\theta) - \int_1^{2} q(\tau)d\tau \right] d\theta - U(2).$$  

At the solution we must have $U(2) = 0$. Integration by parts give us

$$\int_1^{2} \left( \int_1^{2} q(\tau)d\tau \right) d\theta = -\int_1^{2} q(\theta)(1 - \theta)d\theta.$$  

The problem is therefore

$$\max_q \int_1^{2} \left[ \ln q(\theta) - 2\theta q(\theta) + q(\theta) \right] d\theta$$  

subject to the monotonicity constraint. We will first ignore the monotonicity constraint and then show that the solution satisfies it. First order condition for each $\theta$ is given by

$$\frac{1}{q(\theta)} - 2\theta + 1 = 0$$  

and hence

$$q(\theta) = \frac{1}{2\theta - 1}$$  

which is indeed decreasing in $\theta$.  

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2. (30pts.) Consider the signalling game given in Figure 1.

We can find optimal transfers by first calculating $U(\theta)$

$$U(\theta) = \int_0^2 \frac{1}{2\tau - 1} \, d\tau = \frac{1}{2} \ln \frac{3}{2\theta - 1}.$$ 

Therefore,

$$t(\theta) = U(\theta) + \theta q(\theta) = \frac{1}{2} \ln \frac{3}{2\theta - 1} + \frac{\theta}{2\theta - 1}.$$ 

2. (30pts.) Consider the signalling game given in game Figure 1.

(a) (20pts.) Find the set of pure strategy perfect Bayesian equilibria of this game.

Solution

First thing to notice is that $\beta_1(r|Y) = 1$ in any PBE. There are two possibilities regarding $\beta_1(r|X)$:

i. $\beta_1(r|X) = 0$

Bayes’ rule (BR) implies that $\mu(Y|r) = 1, \mu(X|l) = 1$. Therefore, sequential rationality (SR) of player 2 implies that $\beta_2(d|r) = 1, \beta_2(u|l) = 1$. SR of player 1 is satisfied and this assessment is a PBE.

ii. $\beta_1(r|X) = 1$

BR implies that $\mu(X|r) = 4/7$, whereas $\mu(X|l)$ is free. SR of player 2 implies that $\beta_2(d|r) = 1$. SR of player 1 of type $X$ implies that $\beta_2(d|l) = 1$. This, in turn, implies that $\mu(X|l) \leq 4/7$, by SR player 2. Therefore, the following constitute two sets of PBE:

$$(\beta_1(r|Y) = 1, \beta_1(r|X) = 1, \beta_2(u|r) = 1, \beta_2(u|l) = 0, \mu(X|r) = 4/7, \mu(X|l) < 4/7)$$

$$(\beta_1(r|Y) = 1, \beta_1(r|X) = 1, \beta_2(u|r) = 1, \beta_2(u|l) \in (0, 1/3], \mu(X|r) = 4/7, \mu(X|l) = 4/7)$$

iii. (10pts.) Check if the equilibria you find in part (a) satisfy the intuitive criterion.

Solution

The pooling equilibria do not survive the intuitive criterion. In any such equilibrium the payoff of player 1 type $Y$ is equal to 1, whereas the maximum payoff that he can get by playing $l$ is zero, conditional on player 2 playing an undominated action following $l$. On the other hand, equilibrium payoff of type $X$ is 1 and the maximum that he can get by playing $l$, conditional on player 2 playing an undominated action following $l$ is 3. In the notation used in the lecture notes, $J(\beta, \mu, l) = \{Y\}$. Player 2’s best response is $u$, when beliefs are constrained to give positive probability only to type $X$ following $l$, i.e.

$$BR(\Theta_1 \setminus J(\beta, \mu, l), l) = BR(\{X\}, l) = \{u\}.$$ 

This implies that type $X$ would receive a minimum payoff of 3 by playing $l$, which is greater than his equilibrium payoff of 1.

The separating equilibrium, on the other hand, satisfies intuitive criterion. This is because both types are receiving a higher payoff in equilibrium than they could by deviating, irrespective of the way player 2 would respond to such a deviation.