1. Find all the Nash equilibria of the following games.

(a) \[
\begin{array}{c|cc}
T & L & R \\
\hline
B & 2 & 5 \\
L & 3 & 2 \\
B & 4 & 4 \\
\end{array}
\]

Solution
Only NE is \((T, L)\)

(b) \[
\begin{array}{c|cc}
T & L & R \\
\hline
B & 1 & 1 \\
L & 1 & 1 \\
B & 1 & 2 \\
\end{array}
\]

Solution
There are two NE: \((T, L)\), \((B, R)\)

(c) \[
\begin{array}{c|ccc}
T & L & C & R \\
\hline
M & 2 & 3 & 1 & 3 \\
B & 2 & 3 & 2 & 4 \\
B & 1 & 2 & 0 & 5 \\
\end{array}
\]

Solution
There are two NE: \((T, L)\), \((M, C)\)

2. Consider the following game:

(a) Provide values for \(a, b, c, d\) such that the game has a strictly dominant strategy equilibrium.

Solution
For player 1 only \(B\) and for player 2 only \(R\) can be strictly dominant. Any \(c > 3, a < 4\) make \(B\) strictly dominant and any \(b < 1, d > 2\) make \(R\) strictly dominant. So, any \(c > 3, a < 4, b < 1, d > 2\) make \((B, R)\) a strictly dominant strategy equilibrium.

(b) Provide values for \(a, b, c, d\) such that there is no strictly dominant strategy but each player has a strictly dominated strategy.

Solution
Any \(d > 1\) makes \(C\) strictly dominated. If \(b \geq 1\) or \(d \leq 2\), then \(R\) is not strictly dominant. Similarly, any \(c > 1\) makes \(M\) strictly dominated. If \(c \leq 3\) or \(a \geq 4\), then \(B\) is not strictly dominant.
(c) Let $a = 5$ and $b = 2$. Provide values for $c, d$ such that iterated elimination of strictly dominated strategies leads to a unique outcome.

**Solution**

For any $d > 2$ and $c > 1$ IEDS leads to $(B, R)$.

3. Two firms 1 and 2 simultaneously choose time to spend on research $x_1 \geq 0$ and $x_2 \geq 0$. Suppose that firms’ payoff functions are given by

$$u_1(x_1, x_2) = \begin{cases} 10 - x_1, & \text{if } x_1 \geq x_2 \\ -x_1, & \text{if } x_1 < x_2 \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} 10 - x_2, & \text{if } x_2 \geq x_1 \\ -x_2, & \text{if } x_2 < x_1 \end{cases}$$

(a) Show that $x_1 = x_2 = 5$ is a Nash equilibrium.

**Solution**

Let us show that 5 is a best response to 5 for player 1. Playing 5 yields a payoff of 5. Playing $x_1 > 5$ yields $10 - x_1 < 5$, so these are not profitable deviations. Playing $x_1 < 5$ yields $-x_1 \leq 5$, and therefore it is not a profitable deviation. This shows that 5 is a best response to 5 for player 1. The same arguments go for player 2 as well, which proves that $(5,5)$ is a Nash equilibrium.

(b) Find the set of all Nash equilibria. (You may find necessary and sufficient conditions or calculate best response correspondences.)

**Necessary conditions:** In any Nash equilibrium, we must have $x_1 \leq 10$ and $x_2 \leq 10$. Suppose this is not the case, i.e., $x_1 > 10$ or $x_2 > 10$. Then, at least one of the firms obtains negative payoff in equilibrium. But then this firm is not best responding since it can obtain at least zero payoff by choosing 0. Also, in any Nash equilibrium we must have $x_1 = x_2$. Suppose this is not the case; for example, $x_1 > x_2$. In this case Firm 1 is not best responding because it can increase its payoff by choosing $x_2$. Combining these two facts we conclude that in any Nash equilibrium we must have $0 \leq x_1 = x_2 \leq 10$.

**Sufficient conditions:** We claim that any outcome such that $0 \leq x_1 = x_2 \leq 10$ is a Nash equilibrium. In such an outcome, player 1’s payoff is $10 - x_1 \geq 0$. If it chooses $x_1 > x_2$ its payoff decreases to $10 - x_1$ and if it chooses $x_1 < x_2$ its payoff becomes $-x_1 \leq 0$. Therefore, it does not have any profitable deviation, i.e., it is best responding. The same is true for player 2.

We conclude that the set of Nash equilibria is $0 \leq x_1 = x_2 \leq 10$.

The best response correspondences are given by:

$$B_1(x_2) = \begin{cases} 0, & x_2 > 10 \\ \{0, 10\}, & x_2 = 10 \\ x_2, & x_2 < 10 \end{cases}$$

$$B_2(x_1) = \begin{cases} 0, & x_1 > 10 \\ \{0, 10\}, & x_1 = 10 \\ x_2, & x_1 < 10 \end{cases}$$

Plotting the two best response correspondences gives the set of Nash equilibria as $0 \leq x_1 = x_2 \leq 10$. 

$$
\begin{array}{c|cc}
 & L & C & R \\
\hline
T & 3,1 & a,3 & 0,4 \\
M & 1,b & 2,0 & 1,1 \\
B & c,2 & 4,1 & 2,d
\end{array}$$
Figure 1: Player 2’s best response correspondence

Figure 2: Player 1’s best response correspondence

Figure 3: The set of Nash equilibria