1. (30pts.) Find the set of pure strategy Nash equilibria and subgame perfect equilibria of the following game:

![Game Tree Diagram]

Solution

The strategic form of the game is given by the following bimatrix:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F</td>
</tr>
<tr>
<td>OutF</td>
<td>0,20</td>
</tr>
<tr>
<td>OutA</td>
<td>0,20</td>
</tr>
<tr>
<td>InF</td>
<td>-5,-5</td>
</tr>
<tr>
<td>InA</td>
<td>-1,8</td>
</tr>
</tbody>
</table>

The set of pure strategy Nash equilibria are

\[\{(OutF,F), (OutA,F), (InA,A)\}\]

To find the set of SPE, let us first find the set of Nash equilibria of the subgame that starts after In. The strategic form of this game is given by the following bimatrix:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>-5,-5</td>
</tr>
<tr>
<td>A</td>
<td>-1,8</td>
</tr>
</tbody>
</table>
This game has a unique pure strategy Nash equilibrium: \((A, A)\). Given that, Player 1’s optimal choice is \(In\) in equilibrium. Therefore, the unique pure strategy SPE is given by
\[(InA, A)\]

2. (40pts.) Two partners are trying to dissolve their partnership. Player 1 currently owns a share \(s \in (0, 1)\) of the partnership and player 2 owns \(1 - s\). The value of owning the whole partnership is \(v_i \in [0, 1]\) for player \(i = 1, 2\). Player 1 moves first by choosing a price \(p \geq 0\). Player 2 can either buy player 1’s share at \(ps\), in which case the payoffs of player 1 and 2 are \(ps\) and \(v_2 - ps\), respectively, or he may sell his share at \(p(1-s)\), in which case the payoffs are \(v_1 - p(1-s)\) and \(p(1-s)\). The game can be depicted as in Figure 1.

![Figure 1: Partnership Game](image)

(a) (15pts.) First assume that \(v_1 = v_2 = v\), and that this is common knowledge. What is the subgame perfect equilibrium strategy of player 1?

**Solution**

We first show that \(p = v\) in any SPE. Suppose, for contradiction, that \(p < v\) in a SPE. Then, player 2 chooses to buy, which brings a payoff of \(ps\) to player 1. If player 1 offers \(v + \varepsilon\), where
\[
0 < \varepsilon < \frac{s(v - p)}{1 - s}
\]
player 2 would sell and this would bring player 1 a payoff that is strictly greater than \(ps\). Therefore, \(p < v\) cannot be a SPE strategy.

Suppose now that \(p > v\) in a SPE. At this price player 2 chooses to sell, which brings player 1 a payoff of \(v - p(1-s)\). If player 1 offers \(v - \varepsilon\), where
\[
0 < \varepsilon < \min\{v, \frac{(1-s)(p-v)}{s}\}
\]
player 2 would buy and this would bring player 1 a strictly higher payoff.

Therefore, in any SPE \(p = v\). At this price player 2 is indifferent between buying and selling, and whatever he does player 1 gets \(vs\). If player 1 instead offers \(p < v\), her payoff is \(ps < vs\) and if she offers \(p > v\), her payoff is \(v - p(1-s) < vs\). Therefore, offering \(p = v\) is optimal.

(b) (15pts.) Now assume that \(v_i\) is known only to player \(i = 1, 2\), but it is common knowledge that each \(v_i\) is independently and uniformly distributed over \([0, 1]\). What is the equilibrium strategy of player 1?

**Solution**

Expected payoff of player 1 to any price offer \(p\) is given by
\[
\text{prob}(v_2 > p)ps + \text{prob}(v_2 < p)(v_1 - p(1-s)) = (1-p)ps + p(v_1 - p(1-s))
\]
\[
= p(v_1 + s - p).
\]
If the maximum occurs at \( p > 0 \), then the first order condition must hold:

\[ v_1 + s - 2p = 0 \]

or

\[ p = \frac{v_1 + s}{2}. \]

At this price the payoff is strictly positive, whereas the payoff to \( p = 0 \) is zero. Therefore, this is the PBE strategy of player 1.

(c) \( (10\text{pts.}) \) Assume that the value of the partnership is the same for both players, i.e., \( v_1 = v_2 = v \). However, \( v \) is known only to player 2. Player 1 believes that \( v \) has a uniform distribution over \([0,1]\) and all of this is common knowledge. What is the equilibrium price offer of player 1?

Solution

Expected payoff of player 1 to any price offer \( p \) is given by

\[
\text{prob}(v > p)ps + \text{prob}(v < p)(E[v|v < p] - p(1 - s)) = (1 - p)ps + p\left(\frac{p}{2} - p(1 - s)\right)
\]

\[
= p(s - \frac{p}{2}).
\]

If the maximum is in the interior, then the first order condition must hold:

\[ s - p = 0 \]

or \( p = s \). The payoff to this price is positive whereas the payoff to \( p = 0 \) is zero. Therefore, the PBE strategy of player 1 is \( p = s \).

3. \( (30\text{pts.}) \) Find the set of pure strategy perfect Bayesian equilibria of the game given in Figure 2 (Chance chooses \( X \) and \( Y \) with equal probabilities).

\[ \begin{align*}
1, -1 & \quad u & \quad 2, -1 \\
3, 0 & \quad d & \quad 4, 0 \\
-1, 2 & \quad u & \quad 2, 2 \\
1, 0 & \quad d & \quad 4, 0 \\
\end{align*} \]

\[ \begin{align*}
L & \quad 1 & \quad R \\
X & \quad \langle 1/2 \rangle & \quad \langle 1/2 \rangle \\
\text{Chance} & \quad 2 & \quad 2 \\
\end{align*} \]

Figure 2: A Signaling Game

Solution

First notice that \( R \) is strictly dominant for type \( Y \). Therefore, she must be playing \( R \) in any PBE. For type \( X \) there are two possibilities:

(a) Type \( X \) plays \( L \)

This implies that player 2 assigns probability 1 to type \( X \) after \( L \) and type \( Y \) after \( R \). Therefore, he chooses \( d \) after \( L \) and \( u \) after \( R \). Given this strategy by player 2, neither type of player 1 has an incentive to deviate and hence this is a PBE.

(b) Type \( X \) plays \( R \)

This implies that player 2 assigns equal probabilities to \( X \) and \( Y \) after \( R \). Therefore, he chooses \( u \) after \( R \). For this to be an equilibrium, player 2 must be playing \( u \) after \( L \). This, in turn, requires that the probability he assigns to \( X \) after \( L \) is at most \( 2/3 \). These strategies and beliefs constitute a PBE.