Game Theory
Strategic Form Games with Incomplete Information

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Some players have incomplete information about some components of the game:
- Firm does not know rival’s cost
- Bidder does not know valuations of other bidders in an auction

We could also say some players have private information:

What difference does it make?

Suppose you make an offer to buy out a company:

If the value of the company is $V$ it is worth $1.5V$ to you.

The seller accepts only if the offer is at least $V$.

If you know $V$ what do you offer?

You know only that $V$ is uniformly distributed over $[0,100]$. What should you offer?

Enter your name and your bid.

Click here for the EXCEL file.
Bayesian Games

- We will first look at incomplete information games where players move simultaneously
  - Bayesian games
- Later on we will study dynamic games of incomplete information

What is new in a Bayesian game?

- Each player has a type: summarizes a player’s private information
  - Type set for player $i$: $\Theta_i$
    - A generic type: $\theta_i$
  - Set of type profiles: $\Theta = \times_{i \in N} \Theta_i$
    - A generic type profile: $\theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$
- Each player has beliefs about others’ types
  - $p_i : \Theta_i \to \Delta (\Theta_{-i})$
  - $p_i(\theta_{-i}|\theta_i)$
- Players’ payoffs depend on types
  - $u_i : A \times \Theta \to \mathbb{R}$
  - $u_i(a|\theta)$
- Different types of same player may play different strategies
  - $a_i : \Theta_i \to A_i$
  - $\alpha_i : \Theta_i \to \Delta (A_i)$
Bayesian Games

- Incomplete information can be anything about the game
  - Payoff functions
  - Actions available to others
  - Beliefs of others; beliefs of others’ beliefs of others’...

- Harsanyi showed that introducing types in payoffs is adequate
Bayesian equilibrium is a collection of strategies (one for each type of each player) such that each type best responds given her beliefs about other players’ types and their strategies.

Also known as Bayesian Nash or Bayes Nash equilibrium.
Bank Runs

- You (player 1) and another investor (player 2) have a deposit of $100 each in a bank.
- If the bank manager is a good investor you will each get $150 at the end of the year. If not you lose your money.
- You can try to withdraw your money now but the bank has only $100 cash.
  - If only one tries to withdraw she gets $100.
  - If both try to withdraw they each can get $50.
- You believe that the manager is good with probability $q$.
- Player 2 knows whether the manager is good or bad.
- You and player 2 simultaneously decide whether to withdraw or not.
Bank Runs

The payoffs can be summarized as follows

<table>
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<tr>
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<th>W</th>
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<tbody>
<tr>
<td>W</td>
<td>50, 50</td>
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<tr>
<td>N</td>
<td>0, 100</td>
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Good $q$

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Bad $(1 - q)$

Two Possible Types of Bayesian Equilibria

1. Separating Equilibria: Each type plays a different strategy
2. Pooling Equilibria: Each type plays the same strategy

- How would you play if you were Player 2 who knew the banker was bad?
- Player 2 always withdraws in bad state
Separating Equilibria

\[
\begin{array}{c|cc}
 & W & N \\
\hline
W & 50,50 & 100,0 \\
N & 0,100 & 150,150 \\
\end{array}
\]

\[
\begin{array}{c|cc}
 & W & N \\
\hline
W & 50,50 & 100,0 \\
N & 0,100 & 0,0 \\
\end{array}
\]

Good \( q \)

Bad \( (1-q) \)

1. (Good: W, Bad: N)
   - Not possible since W is a dominant strategy for Bad

2. (Good: N, Bad: W)
   Player 1’s expected payoffs
   \[
   \begin{align*}
   & W: \quad q \times 100 + (1-q) \times 50 \\
   & N: \quad q \times 150 + (1-q) \times 0
   \end{align*}
   \]
   Two possibilities
   \[\text{2.1 } q < \frac{1}{2}: \text{ Player 1 chooses W. But then player 2 of Good type must play W, which contradicts our hypothesis that he plays N}\]
   \[\text{2.2 } q \geq \frac{1}{2}: \text{ Player 1 chooses N. The best response of Player 2 of Good type is N, which is the same as our hypothesis}\]

Separating Equilibrium

\[\begin{align*}
& q < \frac{1}{2}: \text{ No separating equilibrium} \\
& q \geq \frac{1}{2}: \text{ Player 1: N, Player 2: (Good: N, Bad: W)}
\end{align*}\]
Pooling Equilibria

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Bad $(1 - q)$

1. (Good: $N$, Bad: $N$)
   - Not possible since $W$ is a dominant strategy for Bad

2. (Good: $W$, Bad: $W$)
   - Player 1’s expected payoffs
     - $W$: $q \times 50 + (1 - q) \times 50$
     - $N$: $q \times 0 + (1 - q) \times 0$
   - Player 1 chooses $W$. Player 2 of Good type’s best response is $W$.
   - Therefore, for any value of $q$ the following is the unique Pooling Equilibrium

Player 1: $W$, Player 2: (Good: $W$, Bad: $W$)

If $q < 1/2$ the only equilibrium is a bank run
Cournot Duopoly with Incomplete Information about Costs

- Two firms. They choose how much to produce $q_i \in \mathbb{R}_+$
- Firm 1 has high cost: $c_H$
- Firm 2 has either low or high cost: $c_L$ or $c_H$
- Firm 1 believes that Firm 2 has low cost with probability $\mu \in [0, 1]$
- payoff function of player $i$ with cost $c_j$

$$u_i(q_1, q_2, c_j) = (a - (q_1 + q_2)) q_i - c_j q_i$$

- Strategies:

$$q_1 \in \mathbb{R}_+ \quad q_2 : \{c_L, c_H\} \to \mathbb{R}_+$$
Complete Information

- Firm 1
  \[
  \max_{q_1} \left( a - (q_1 + q_2) \right) q_1 - c_H q_1
  \]
  
- Best response correspondence
  \[
  BR_1(q_2) = \frac{a - q_2 - c_H}{2}
  \]

- Firm 2
  \[
  \max_{q_2} \left( a - (q_1 + q_2) \right) q_2 - c_j q_2
  \]

- Best response correspondences
  \[
  BR_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}
  \]
  \[
  BR_2(q_1, c_H) = \frac{a - q_1 - c_H}{2}
  \]
Nash Equilibrium

- If Firm 2’s cost is $c_H$
  
  \[ q_1 = q_2 = \frac{a - c_H}{3} \]

- If Firm 2’s cost is $c_L$
  
  \[ q_1 = \frac{a - c_H - (c_H - c_L)}{3} \]

  \[ q_2 = \frac{a - c_H + (c_H - c_L)}{3} \]
Incomplete Information

- Firm 2

\[
\max_{q_2} \left( a - (q_1 + q_2) \right) q_2 - c_j q_2
\]

- Best response correspondences

\[
BR_2(q_1, c_L) = \frac{a - q_1 - c_L}{2}
\]

\[
BR_2(q_1, c_H) = \frac{a - q_1 - c_H}{2}
\]

- Firm 1 maximizes

\[
\mu \{ [a - (q_1 + q_2(c_L))] q_1 - c_H q_1 \} + (1 - \mu) \{ [a - (q_1 + q_2(c_H))] q_1 - c_H q_1 \}
\]

- Best response correspondence

\[
BR_1(q_2(c_L), q_2(c_H)) = \frac{a - [\mu q_2(c_L) + (1 - \mu) q_2(c_H)] - c_H}{2}
\]
Bayesian Equilibrium

\[ q_1 = \frac{a - c_H - \mu(c_H - c_L)}{3} \]

\[ q_2(c_L) = \frac{a - c_L + (c_H - c_L)}{3} - (1 - \mu) \frac{c_H - c_L}{6} \]

\[ q_2(c_H) = \frac{a - c_H}{3} + \mu \frac{c_H - c_L}{6} \]

• Is information good or bad for Firm 1?
• Does Firm 2 want Firm 1 to know its costs?
Complete vs. Incomplete Information

Complete Information

Incomplete Information

\[
q_2 \quad q_1
\]

\[
\frac{a-c_H}{2} \quad \frac{a-c_L}{2} \quad a-c_H \quad a-c_L
\]

\[
BR_1(q_2) \quad BR_2(q_1, c_L) \quad BR_2(q_1, c_H)
\]

\[
q_2(c_L) \quad E[q_2] \quad q_2(c_H)
\]