1. (20 pts.) Consider the strategic form game represented by the following bimatrix.

\[
\begin{array}{c|cc}
& S & H \\
\hline
S & 4 & 0 \\
H & 2 & 2 \\
N & 3 & 1 \\
\end{array}
\]

(a) (10 pts) Does any player have a strictly dominated action?

Solution

An action is strictly dominated if and only if it is a never best response. Player 2 does not have a strictly dominated action because \( S \) is a best response to the belief that puts probability one on \( S \) and \( H \) is a best response to the belief that puts probability one on \( H \).

To check for player 1, let \( q \) be the probability assigned to the event that player 2 plays \( S \). Then, expected payoffs to the three actions are given by

\[
\begin{align*}
U_1(S, q) &= 4q \\
U_1(H, q) &= 2 \\
U_1(N, q) &= 2q + 1
\end{align*}
\]

For \( q > 1/2 \), unique best response is \( S \) and for \( q < 1/2 \) unique best response is \( H \). When \( q = 1/2 \), all three are best responses. (see Figure below). Therefore, all three actions are best responses for some beliefs, which implies that player 1 does not have strictly dominated action either.

(b) (10 pts) Find the set of all Nash equilibria (pure and mixed).

Solution

Let \( q \) be the probability with which player 2 plays \( S \). There are three possibilities:

i. \( q < 1/2 \). Player 1’s unique best response is \( H \), to which player 2’s best response is \( H \), or \( q = 0 \). Therefore, one Nash equilibrium is \( (H, H) \).

ii. \( q > 1/2 \). Player 1’s unique best response is \( S \), to which player 2’s best response is \( S \), or \( q = 1 \). Therefore, another Nash equilibrium is \( (S, S) \).

iii. \( q = 1/2 \). All three actions as well as any mixed strategy are best responses for player 1. \( q = 1/2 \) implies that the expected payoff of player 2 to \( S \) and \( H \) must be the same. Letting \( \text{prob}(S) = p_1 \) and \( \text{prob}(H) = p_2 \) for player 1, this implies that

\[ 4p_1 = 2p_1 + 2p_2, \]

\[ p_1 = p_2. \]
or \( p_1 = p_2 = p \). This implies that \( \text{prob}(N) = 1 - 2p \) and hence we must have \( 0 \leq 2p - 1 \leq 1 \) or \( 0 \leq p \leq 1/2 \). Therefore, for any \( p \in [0, 1/2] \) there is a Nash equilibrium in which player 1 plays \( \text{prob}(S) = \text{prob}(H) = p \), \( \text{prob}(N) = 1 - 2p \)

and player 2 plays \( \text{prob}(S) = 1/2 \).

2. (20 pts) A buyer and a seller simultaneously submit a price, which can be any non-negative number. If the price chosen by the buyer is at least as large as the price chosen by the seller, i.e., \( p_b \geq p_s \), trade occurs and the buyer pays \( p_b \) whereas the seller receives \( p_s \). The rest, i.e., \( p_b - p_s \), goes to a charity. If trade occurs, payoff of the buyer is his value \( v \) minus the price he pays and the payoff of the seller is the price she receives minus her cost, \( c \). Assume that \( v > c \geq 0 \). If trade does not occur, both players receive zero payoff. Therefore, the payoff functions are given by

\[
\begin{align*}
\text{u}_b(p_b, p_s) &= \begin{cases} 
0, & p_b < p_s \\
 v - p_b, & p_b \geq p_s
\end{cases} \\
\text{u}_s(p_b, p_s) &= \begin{cases} 
0, & p_b < p_s \\
 p_s - c, & p_b \geq p_s
\end{cases}
\end{align*}
\]

(a) (10 pts) Show that any \( (p_b, p_s) \) such that \( c \leq p_b = p_s \leq v \) is a Nash equilibrium.

**Solution**

Take any \( p_s \in [c, v] \). Buyer’s payoff to \( p_b = p_s \) is \( v - p_b \geq 0 \). If he instead chooses \( p_b < p_s \) he gets zero. If he chooses \( p_b > p_s \), he gets \( v - p_b < v - p_s \). Therefore, he does not have a unilateral profitable deviation.

Now take any \( p_b \in [c, v] \). Seller’s payoff to \( p_s = p_b \) is \( p_b - c \geq 0 \). If she instead chooses \( p_s > p_b \) she gets zero. If she chooses \( p_s < p_b \), she gets \( p_s - c < p_b - c \). Therefore, she does not have a unilateral profitable deviation either.

(b) (10 pts) Find the set of all pure strategy Nash equilibria.

**Solution**

The best response correspondences of the players are given by:

\[
\begin{align*}
B_b(p_s) &= \begin{cases} 
p_s, & p_s < v \\
[0, p_s], & p_s = v \\
[0, \infty), & p_s > v
\end{cases} \\
B_s(p_b) &= \begin{cases} 
(p_b, \infty), & p_b < c \\
[p_b, \infty), & p_b = c \\
p_b, & p_b > c
\end{cases}
\end{align*}
\]

Figures 1-3 plot the best response correspondences and find where they intersect. As can be seen from the Figure 3, in addition to the Nash equilibria given in part (b), any \( (p_b, p_s) \) such that \( p_b \leq c \) and \( p_s \geq v \) is also a Nash equilibrium.
3. **(30pts.)** An incumbent monopolist firm (player 1) is deciding whether to undertake a costly investment to expand its production capacity. A potential entrant (player 2) is considering entering into this market, which is profitable only if the incumbent does not invest. The cost of investment is either low or high, which is known by the incumbent but not by the entrant. The entrant believes that the entrant’s cost is low with probability \( q \in (0, 1) \). Payoffs to different action and type profiles are given below:

\[
\begin{array}{c|cc}
 & E & O \\
\hline
I & 1, -1 & 6, 0 \\
N & 2, 2 & 4, 0 \\
\end{array}
\quad \begin{array}{c|cc}
 & E & O \\
\hline
I & 0, -1 & 2, 0 \\
N & 2, 2 & 4, 0 \\
\end{array}
\]

where the row player is the incumbent and the column player is the entrant; \( I \) denotes invest, \( N \) not invest, \( E \) enter, and \( O \) stay out. All of this is common knowledge.

(a) **(10 pts)** Find the set of pooling pure strategy Bayesian equilibria.

**Solution**

Let \( \sigma_i \) denote player \( i \)'s strategy. Note that for high cost player 1, \( N \) is a strictly dominant strategy. Therefore, in any Bayesian equilibrium he plays \( N \). The only candidate for a pooling strategy equilibrium is therefore for both types to play \( N \). The best response of player 2 to this is \( E \), to which the best response of each type of player 1 is indeed \( N \). Therefore,

\[
\sigma_1(L) = \sigma_1(H) = N, \sigma_2 = E
\]

is a pooling Bayesian equilibrium.

(b) **(10 pts)** Find the set of separating pure strategy Bayesian equilibria.

**Solution**

The only candidate is \( \sigma_1(L) = I, \sigma_1(H) = N \). For the low cost to invest, player 2 must be choosing to stay out. This implies

\[
0 \geq q \times (-1) + (1 - q) \times 2
\]

or \( q \geq 2/3 \). Therefore, the following is a Bayesian equilibrium if and only if \( q \geq 2/3 \):

\[
\sigma_1(L) = I, \sigma_1(H) = N, \sigma_2 = O.
\]

(c) **(10 pts)** Is there an equilibrium in which at least one type of player 1 chooses both \( I \) and \( N \) with positive probability?

**Solution**

Only the low cost could be completely mixing. So, let \( \sigma_1(I|L) \in (0, 1) \). This implies that expected payoffs to \( I \) and \( N \) are equal:

\[
\sigma_2(E) \times 1 + (1 - \sigma_2(E)) \times 6 = \sigma_2(E) \times 2 + (1 - \sigma_2(E)) \times 4
\]

which is solved as

\[
\sigma_2(E) = \frac{2}{3}
\]

\( \sigma_2(E) \in (0, 1) \) implies that player 2 must be indifferent between \( E \) and \( O \), i.e.,

\[
q \{\sigma_1(I|L) \times (-1) + (1 - \sigma_1(I|L)) \times 2\} + (1 - q) \times 2 = 0
\]

which is solved as

\[
\sigma_1(I|L) = \frac{2}{3q}
\]

This is in \( (0, 1) \) if and only if

\[
q > \frac{2}{3}
\]

Therefore, there is such an equilibrium if \( q > 2/3 \).
4. (30pts.) Two players, who are involved in a competition to win a prize, are considering how much to bribe a bureaucrat, who will award the prize to the highest briber. In case of a tie, each player wins the prize with equal probability. They choose how much to bribe independently and without knowing the choice of the other player, and if they don’t win the prize they cannot get the bribe back. The value of the prize to player $i$ is $v_i$. Player $i \neq j$ knows $v_i$ but is uncertain about $v_j$: she believes that $v_j$ is a random variable with uniform distribution over $[0, 1]$. Therefore, for any bribe profile $(b_1, b_2)$, the payoff function of player $i$ is:

$$u_i(b_1, b_2) = \begin{cases} 
-b_i, & \text{if } b_i < b_j \\
\frac{1}{2}v_i - b_i, & \text{if } b_i = b_j \\
v_i - b_i, & \text{if } b_i > b_j 
\end{cases}$$

where $j \neq i$. All of this is common knowledge.

Denote the pure strategy of player $i$ of type $v$ by $\beta_i(v)$. Find a pure strategy Bayesian equilibrium of this game, where $\beta_1(v) = \beta_2(v) = \beta(v)$ for all $v$. You may assume that $\beta$ is strictly increasing and differentiable.

Inverse Function Theorem:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

**Solution**

In equilibrium we must have $\beta(0) = 0$, since otherwise the player receives a negative payoff, whereas bribing zero yields zero payoff. Suppose now that player 2 follows the strategy $\beta$. Then, the expected payoff of player 1 of type $v$ who chooses $b$ is given by

$$\text{prob}(b > \beta(V_2))v - b = \text{prob}(V_2 < \beta^{-1}(b))v - b$$

It is never optimal to choose $b > \beta(1)$, since by lowering $b$ the player can increase payoff. Therefore, $b \leq \beta(1)$. Also, $\beta(1)$ is strictly greater than $\beta(0)$. Therefore, $0 \leq \beta^{-1}(b) \leq 1$. This implies that $\text{prob}(V_2 < \beta^{-1}(b)) = \beta^{-1}(b)$, and hence the payoff function is given by

$$\beta^{-1}(b)v - b.$$

Since $\beta$ is strictly increasing $\beta(v) > 0$ for all $v > 0$ and therefore the FOC must hold with equality for all $v > 0$:

$$\frac{1}{\beta'(\beta^{-1}(b))}v - 1 = 0$$

Substituting $b = \beta(v_1)$ we get

$$\beta'(v) = v$$

for all $v > 0$. By the fundamental theorem of calculus and the fact that we must have $\beta(0) = 0$ the equilibrium strategy must be

$$\beta(v) = \int_0^v xdx = \frac{v^2}{2}$$

(2)

It can be easily verified that $\beta$ is strictly increasing and $\beta(0) = 0$.

So, we have shown that any symmetric Bayesian equilibrium with strictly increasing strategies must take the form given in (2). We now have to show that it is indeed an equilibrium. Consider player 1 with type $v$ and note that we only need to consider deviations in the range of $\beta$. This is because if a deviation strictly greater than $\beta(1)$ is profitable, then so will be a deviation to $\beta(1)$. Any deviation must clearly be greater than or equal to $\beta(0) = 0$.

Let $U_i(y|v)$ denote the expected payoff of player $i$ with type $v$ who plays $\beta(y)$ when the other player plays according to $\beta$ as given in (2). We have to show that $U_i(y|v) \geq U_i(y|v)$ for all $y$. We can calculate $U_i(y|v)$ as follows:

$$U_i(y|v) = \text{prob}(\beta(y) > \beta(V_2))v - \beta(y) = yv - \beta(y)$$

Therefore, $U_i(y|v) - U_i(y|v) = \frac{v^2}{2} - yv + \frac{y^2}{2}$

$$\frac{1}{2}(v - y)^2 \geq 0$$

and we are done.