1. (30pts.) Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>3,1</td>
<td>a,3</td>
<td>0.4</td>
</tr>
<tr>
<td>M</td>
<td>1,b</td>
<td>2,0</td>
<td>1.1</td>
</tr>
<tr>
<td>B</td>
<td>c,2</td>
<td>4,1</td>
<td>2,d</td>
</tr>
</tbody>
</table>

(a) (10 pts) Assume $a = 5, b = 2, c = 4, d = 2$ and find the set of (pure and mixed) Nash equilibria of this game.

**Solution**

$M$ and $C$ are strictly dominated and hence cannot receive positive probability in any Nash equilibrium. Given that only $L$ and $R$ receive positive probability, $T$ cannot receive positive probability either. So, in any Nash equilibrium player 1 must play $B$ with probability one. Given that, any probability distribution over $L$ and $R$ is a best response for player 2. In other words, the set of Nash equilibria is given by

$$\{(B, q \times L \oplus (1 - q) \times R), q \in [0, 1]\}$$

(b) (10 pts) Provide values for $a, b, c, d$ such that the game has a strictly dominant strategy equilibrium.

**Solution**

For player 1 only $B$ and for player 2 only $R$ can be strictly dominant. Any $c > 3, a < 4$ make $B$ strictly dominant and any $b < 1, d > 2$ make $R$ strictly dominant. So, any $c > 3, a < 4, b < 1, d > 2$ make $(B, R)$ a strictly dominant strategy equilibrium.

(c) (10 pts) Provide values for $a, b, c, d$ such that there is no strictly dominant strategy equilibrium but there is a unique pure strategy Nash equilibrium.

**Solution**

Only $(B, L)$ and $(B, R)$ can be a pure strategy Nash equilibrium. For $(B, L)$ to be the unique NE, we must have $c > 3$ and $d < 2$. $d < 2$ also guarantees that player 2 does not have a strictly dominant strategy, and hence the game does not have a strictly dominant strategy equilibrium. $b$ and $a$ can be specified arbitrarily.

For $(B, R)$ to be the unique NE, $d > 2$ is sufficient. If $b \geq 1$ or $c \leq 3$ or $a \geq 4$, then there is no strictly dominant strategy equilibrium.

2. (20 pts) Player 1 and 2 are trying to share a cake of size 100. They each simultaneously submit a real number in the interval $[0, 100]$. If the sum of their numbers is smaller than or equal to 100, then each receives the number she submitted. If the sum is greater than 100, then the player who submitted the smaller number, say $x$, receives that number while the other receives $100 - x$. If the sum is greater than 100 and they are equal, then each receives 50.

(a) (10 pts) Find the set of pure strategy Nash equilibria of this game.

**Solution**

The best response correspondence of player $i \neq j$ is given by

$$BR_i(a_j) = \begin{cases} [100 - a_j, 100], & a_j \leq 50, \\ \emptyset, & a_j > 50 \end{cases}$$

Plotting shows that the unique Nash equilibrium is $(50, 50)$. 
(b) (10 pts) Consider the same game but assume that the numbers must be integers, i.e., 0, 1, 2, \ldots, 100. Consider the following order of eliminating weakly dominated strategies: At each stage all the weakly dominated strategies of both players are eliminated. Which outcome does that procedure lead to?

Solution
In the first round, every number \( x \leq 50 \) is weakly dominated by 51. No other action is weakly dominated in the first round, because 100 is a unique best response of 0 and any number \( x \in [52, 99] \) is a best unique response to \( x + 1 \). In the second round 100 is weakly dominated by 51. In the third round, 99 is weakly dominated by 51, and so on. Let \( 52 \leq k \leq 100 \) and player 2’s action \( a_2 \in \{51, 52, \ldots, k\} \). Then, for player 1, \( k \) is weakly dominated by 51:

<table>
<thead>
<tr>
<th>( a_2 = k )</th>
<th>51 &lt; ( a_2 &lt; k )</th>
<th>( a_2 = 51 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>50</td>
<td>100 - ( a_2 )</td>
</tr>
<tr>
<td>51</td>
<td>51</td>
<td>51</td>
</tr>
<tr>
<td>50</td>
<td>49</td>
<td>100</td>
</tr>
</tbody>
</table>

Therefore, the unique outcome that survives this procedure is (51, 51).

3. (20pts.) A robber decides whether to attack (A) or pass (P) and a victim simultaneously decides whether to fight (F) or yield (Y). Victim is either weak or strong and the payoff matrices (where robber is the row player and victim is column player) corresponding to the two types are given as follows:

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>P</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Weak (1 - \( q \))

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>P</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Strong (\( q \))

Robber does not know the type of the victim but believes that she is strong with probability \( q \in (0, 1) \). All of this is common knowledge.

(a) (10 pts) Find the set of pure strategy Bayesian equilibria of this game as a function of \( q \).

Solution
Notice that \( F \) is a strictly dominant strategy for the strong type. Therefore, we may have two possible equilibrium configurations:

(1) Strong plays \( F \), Weak plays \( F \)
Robber’s best response to these strategies is \( P \). Weak victim’s best response to that is \( F \) and therefore the following is a Bayesian equilibrium independent of \( q \):

\[(P, (\text{Strong} \ F, \ \text{Weak} \ F))\]

(2) Strong plays \( F \), Weak plays \( Y \)
Expected payoff of the robber from playing \( A \) is given by

\[(1 - q) \times 2 + q \times (-1) = 2 - 3q\]

whereas the expected payoff to \( P \) is zero. For the weak victim to play \( Y \) robber must be playing \( A \), which implies that

\[2 - 3q \geq 0\]

or

\[q \leq \frac{2}{3}\]

If this holds the following is another Bayesian equilibrium:

\[(A, (\text{Strong} \ F, \ \text{Weak} \ Y))\]

In sum, if \( q > 2/3 \) unique Bayesian equilibrium is given by

\[(P, (\text{Strong} \ F, \ \text{Weak} \ F))\]
and if \( q \leq 2/3 \) there is another Bayesian equilibrium

\[
(A, (F^{'}, Y))
\]

(b) (10 pts) Assume that \( q < 2/3 \) and find a Bayesian equilibrium in which the robber plays both \( A \) and \( P \) with strictly positive probabilities.

Solution

As we observed before in any Bayesian equilibrium the strong type plays \( F \) with probability one. So, let \( \alpha \) be the probability with which the weak type plays \( F \) and \( \beta \) be the probability with which the robber plays \( A \) in equilibrium. For the robber to play both actions with positive probability expected payoff to each action must be equal:

\[
(1 - q)[\alpha \times (-1) + (1 - \alpha) \times (2)] + q \times (-1) = 0
\]

This is solved as

\[
\alpha = \frac{2 - 3q}{3 - 3q}
\]

Since \( q < 2/3 \), \( \alpha \in (0, 1) \), which implies that the expected payoffs to \( F \) and \( Y \) must be the same for the weak type:

\[
\beta \times (-3) + (1 - \beta) \times (1) = \beta \times (-2) + (1 - \beta) \times (0)
\]

This is solved as \( \beta = 1/2 \). Therefore, the strategy profile in which strong type plays \( F \) with probability one and

\[
\alpha = \frac{2 - 3q}{3 - 3q}, \quad \beta = \frac{1}{2}
\]

is a Bayesian equilibrium in which the robber plays both \( A \) and \( P \) with strictly positive probabilities.

4. (30pts.) Consider the following two bidder second-price sealed bid auction. Players receive private and independent signals, \( t_1 \) and \( t_2 \), that have a uniform distribution over \([0, 1]\). Value of the object to player \( i \) is \( \alpha t_i + \gamma t_j \), where \( i \neq j \) and \( \alpha > \gamma \geq 0 \).

Note: Inverse Function Theorem:

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}
\]

Leibniz integral rule:

\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(y, x)dy = \frac{db(x)}{dx} f(b(x), x) - \frac{da(x)}{dx} f(a(x), x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(y, x)dy
\]

(a) (15 pts) Show that \( \beta_i(t_i) = (\alpha + \gamma)t_i, i = 1, 2 \) is a Bayesian equilibrium of this game.

Solution

Expected payoff of player 1 who has signal \( t_1 \) and bids \( (\alpha + \gamma)t_1 \) is given by

\[
E[\alpha t_1 + \gamma t_2 - (\alpha + \gamma)t_2 | (\alpha + \gamma)t_1 > (\alpha + \gamma)t_2] \text{prob}(\alpha + \gamma)t_1 > (\alpha + \gamma)t_2
\]

which is equal to

\[
\left(\alpha t_1 - \frac{t_1^2}{2}\right) t_1 = \alpha \frac{t_1^2}{2}
\]

Now consider an arbitrary deviation to \( b \). Since player 2’s bid is at most equal to \( \alpha + \gamma \) we can limit ourselves to deviation is \([0, (\alpha + \gamma)], \) or equivalently to deviations in the form of \((\alpha + \gamma)y\) for some \( y \in [0, 1] \). Expected payoff to any such deviation is given by

\[
\alpha \left(t_1 - \frac{y}{2}\right) y
\]

The difference between these payoffs is

\[
\alpha \frac{t_1^2}{2} - \alpha \left(t_1 - \frac{y}{2}\right) y = \alpha \frac{t_1^2}{2} (t_1 - y)^2 \geq 0.
\]

Therefore, there is no profitable deviation, i.e., \((\alpha + \gamma)t_1 \) is a best response to \((\alpha + \gamma)t_2 \). The same arguments go through for player 2, which shows that these strategies constitute a Bayesian equilibrium.
(b) **(15 pts)** Characterize all symmetric equilibria in which player $i$ bids $\beta(t_i)$, where $\beta$ is strictly increasing and differentiable.

**Solution**

Expected payoff of player 1 when her signal is $t_1$ and bids $b$ is

$$
\int_0^{\beta^{-1}(b)} (\alpha t_1 + \gamma t_2 - \beta(t_2))dt_2
$$

The first order conditions for maximizing this is given by

$$
\frac{1}{\beta'(\beta^{-1}(b))} (\alpha t_1 + \gamma \beta^{-1}(b) - \beta(\beta^{-1}(b))) = 0
$$

This must hold when $b = \beta(t_1)$, which implies that

$$
\frac{1}{\beta'(t_1)} (\alpha t_1 + \gamma t_1 - \beta(t_1)) = 0.
$$

Since $\beta' > 0$, this implies that

$$
\beta(t_1) = (\alpha + \gamma) t_1.
$$

In part (a) we have shown that $(\alpha + \gamma) t_i$ is a best response to $(\alpha + \gamma) t_j$, $i \neq j$. Therefore, the unique symmetric equilibrium in strictly increasing and differentiable strategies is given by

$$
\beta(t_i) = (\alpha + \gamma) t_i, i = 1, 2.
$$