Microeconomics II
Extensive Form Games II

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Extensive Form Games with Imperfect Information

- In extensive form games with perfect information every player observes the previous moves made by all the players.
- What happens if some of the previous moves are not observed?
- We cannot apply backward induction algorithm anymore.

Consider the following modified entry game:

Player 1 doesn’t know whether Player 2 fights or accommodates.

We cannot determine the optimal action for Player 1 at that information set...

and therefore cannot apply backward induction algorithm.

Subgame Perfect Equilibrium

We will introduce another solution concept: Subgame Perfect Equilibrium.

- For any extensive form game \( \Gamma = (N, H, \iota, \beta_c, I, (u_i)) \) and history \( h \in H \setminus Z \) define the restriction of \( H \) to histories that start with \( h \) as follows:

\[
H|_h = \{h\} \cup \{h'' \in H : \exists h' \text{ s.t. } h'' = (h, h')\}
\]

\( H|_h \) contains all the histories in \( \Gamma \) that start with \( h \).

- We similarly define \( N|_h, \iota|_h, \beta_c|_h, I|_h, (u_i)|_h \).

Definition (Subgame)

Let \( \Gamma = (N, H, \iota, \beta_c, I, (u_i)) \) be an extensive form game and take any \( h \in H \setminus Z \). We say that \( \Gamma|_h = (N|_h, H|_h, \iota|_h, \beta_c|_h, I|_h, (u_i)|_h) \) is a subgame of \( \Gamma \) if:

- it starts at a singleton information set: \( I(h) \) is singleton.
- it contains every history that follows \( h \). If there exists \( h' \in H \) such that \( (h', h, h'') \in H \) then \( (h, h'') \in H|_h \).
- whenever it contains a node in an information set, it contains all the nodes in that information set. If \( h' \in H|_h \), then \( I(h') \subseteq H|_h \).

- The game itself is a subgame.
- We call subgames other than the game itself proper subgames.
Subgame Perfect Equilibrium

**Definition**

A strategy profile in an extensive form game is a subgame perfect equilibrium (SPE) if it induces a Nash equilibrium in every subgame of the game. More formally, \( s^* \in S \) is a SPE of \( \Gamma \) if \( s^*|_h \in S|h \) is a Nash equilibrium of \( \Gamma|h \) for all \( h \in H \setminus Z \).

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Subgame Perfect Equilibrium

Consider the following game

- There are two subgames: the game itself and the one after history \( E \).
- Strategic form of the game

\[
\begin{array}{c|cc}
& F & A \\
\hline
O & 0, 20 & 0, 20 \\
F & -5, -5 & 8, -1 \\
E & -1, 8 & 10, 10 \\
A & & \\
\end{array}
\]

- Set of Nash equilibria

\[ N(\Gamma) = \{(OF, F), (OA, F), (EA, A)\} \]
Subgame Perfect Equilibrium

In practice you may use an algorithm similar to backward induction:

- Find the Nash equilibria of the “smallest” subgame(s)
- Fix one for each subgame and attach payoffs to its initial node
- Repeat with the reduced game

Subgame Perfect Equilibrium

The “smallest” subgame

Its strategic form

Unique Nash equilibrium of the subgame is \((A, A)\)

Reduced subgame is

Its unique Nash equilibrium is \((E)\)

Therefore the unique SPE of the game is \(((E, A), A)\)

One-Deviation Property

- In complicated extensive form games checking whether a strategy profile is a SPE could be quite difficult.
- One-deviation property simplifies this process tremendously.

Definition

Let \(\Gamma\) be an extensive form game with perfect information. For any \(s_i|_h \in S_i|_h\) in a subgame \(\Gamma|_h\) and an action \(a_i \in A(h)\) such that \(a_i \neq s_i|_h(h)\) define a one-deviation (or one-shot deviation) \(s_{i|_h}|_{h'}\) as follows:

\[
s_{i|_h}|_{h'}(h') = \begin{cases} a_i, & h' = h \\ s_i|_h(h'), & \text{otherwise} \end{cases}
\]
One-Deviation Property

Proposition (One-Deviation Property)
Let \( \Gamma \) be a finite horizon extensive form game with perfect information. A strategy profile \( s^* \) is a SPE of \( \Gamma \) if, and only if, for every player \( i \in N \) and every \( h \in H \setminus Z \) for which \( i(h) = i \) we have
\[
U_i|h(s^*_i|h, s^*_{-i}|h) \geq U_i|h(s^0_i|h, s^*_{-i}|h)
\]
for every one-deviation \( s^0_i|h \).

There are versions of ODP for games with imperfect information and infinite horizon games.

One-Deviation Property: Example
Consider the following (centipede) game
\[
\begin{array}{c|cc|cc|cc|c}
 & C & 2 & C & 2 & C & 1 & 1,6 \\
\hline
S & 1,0 & 0,2 & 3,1 & 2,4 & 5,3 & \\
\end{array}
\]
Is \( s_1 = SSS, s_2 = SS \) a SPE?
- Is \( SSS \) a best response to \( SS \)?
  - Possible deviations: \( SSC, SCS, SCC, CSS, CSC, CCS, CCC \)
  - ODP says you only need to check against \( CSS \)
    * \( SSS \rightarrow 1 \)
    * \( CSS \rightarrow 0 \)
- Is \( SS \) a best response to \( S \)?
  - Possible deviations: \( SC, CS, CC \)
  - ODP says you only need to check against \( CS \)
    * \( SS \rightarrow 3 \)
    * \( CS \rightarrow 2 \)
- Is \( S \) a best response?

Subgame Perfect Equilibrium

What is the pure strategy SPE of the following game?

\[
\begin{array}{c|c|c}
1 & N & 2 \\
\hline
N & -1/2, -1/2 & \text{---} \\
M & 1 & \text{---} \\
\hline
H & \text{---} & T \\
T & \text{---} & T \\
\hline
1, -1 & -1, 1 & -1, 1 & 1, -1
\end{array}
\]

We have to introduce mixed strategies.

Mixed vs. Behavioral Strategies
There are two ways through which a player may randomize:
- randomize over pure strategies in the entire game: mixed strategy
- randomize separately at each information set: behavioral strategy

For any \( i \in N \) a mixed strategy \( \sigma_i \) is a probability distribution over \( S_i \).

The set of mixed strategies for player \( i \)
\[
\Sigma_i = \Delta(\times_{I \in \mathcal{I}_i}A(I))
\]

\( \sigma_i(s_i) \): probability assigned to \( s_i \)

A behavioral strategy of player \( i \in N \) is a collection of independent probability distributions \( \beta_i = (\beta_i(\cdot|I))_{I \in \mathcal{I}_i} \), where \( \beta_i(\cdot|I) \) is a probability distribution over \( A(I) \).

The set of behavioral strategies for player \( i \)
\[
\mathcal{B}_i = \times_{I \in \mathcal{I}_i} \Delta(A(I))
\]

\( \beta_i(a|I) \) denotes the probability assigned to \( a \in A(I) \).
Mixed vs. Behavioral Strategies

- A mixed strategy:
  \[ \sigma_1(NH) = \frac{1}{2} \]
  \[ \sigma_1(MH) = \frac{1}{2} \]

- A behavioral strategy:
  \[ \beta_1(N|\{a_0\}) = \frac{1}{2} \]
  \[ \beta_1(H|\{(M, H), (M, T)\}) = \frac{1}{2} \]

Strategies and Outcomes

A mixed strategy profile \( \sigma \in \Sigma \) and Nature’s probability distribution \( \beta_c \) induces an outcome \( P^\sigma \).

- A probability distribution over the set of terminal histories \( Z \)
- Let \( P^\sigma(z) \) denote the probability of \( z \in Z \) under \( \sigma \)

For any \( i \in N \cup \{c\}, I \in \mathcal{I}_i, \) and \( h \in I \) define

\[ s_i(h) = s_i(I) \]
\[ \beta_i(.)|h = \beta_i(.)|I \]

Take any non-empty history \( h = (a_0, a_1, \ldots, a_k) \). Fix any \( i \in N \cup \{c\} \) and \( s_i \in S_i \). We say that \( s_i \) is consistent with \( h \) if for all \( h_l = (a_0, a_1, \ldots, a_l), l < k, \) for which \( i(h_l) = i \)

\[ s_i(h_l) = a_{l+1} \]

Note that every \( s_i \) is consistent with \( a_0 \) and any history \( h \) in which \( i \) does not move.

Strategies and Outcomes

Similarly for a behavioral strategy profile \( \beta \in B \) we can define an outcome \( P^\beta \).

- Take any non-empty history \( h = (a_0, a_1, \ldots, a_k) \) and a behavioral strategy profile \( \beta \). For any \( l < k, \) let \( h_l = (a_0, a_1, \ldots, a_l) \) and define

\[ P^\beta(h_l) = \prod_{i=0}^{k-1} \beta_i(h_l)(a_{l+1}|h_l) \]

- We assume that players’ preferences can be represented by expected payoffs:

\[ U_i(\sigma) = \sum_{z \in Z} P^\sigma(z) u_i(z) \]
\[ U_i(\beta) = \sum_{z \in Z} P^\beta(z) u_i(z) \]
Strategies and Outcomes: Example

Suppose the mixed strategy profile is
\[ \sigma_1(NH) = 1/2, \sigma_1(MH) = 1/2, \sigma_2(H) = 1/3 \]
and we want to calculate
\[ P^\sigma(M, T, H) \]

Pure strategies of player 1 that are consistent with \((M, T, H)\) must have
\[ s_1(a_0) = M, s_1(M, T) = H \]

Therefore
\[ S_1^{(M,T,H)} = \{MH\} \]
\[ P^{s_1}(M, T, H) = \sum_{s_1 \in S_1^{(M,T,H)}} \sigma_1(s_1) = \sigma_1(MH) = 1/2 \]

Strategies and Outcomes: Example

Do the same with player 2. We have
\[ S_2^{(M,T,H)} = \{T\} \]
\[ P^{s_2}(M, T, H) = \sum_{s_2 \in S_2^{(M,T,H)}} \sigma_2(s_2) = \sigma_2(T) = 2/3 \]

So,
\[ P^\sigma(M, T, H) = \prod_{i \in \{1,2\}} P^{s_i}(M, T, H) = 1/3 \]

Similarly,
\[ P^{s_1}(N) = \sigma_1(NH) + \sigma_1(NT) = 1/2 \]
\[ P^{s_2}(N) = 1 \]
\[ P^\sigma(N) = 1/2 \]

Verify that:
\[ P^\sigma(M, H, H) = 1/6, \quad P^\sigma(M, H, T) = P^\sigma(M, T, T) = 0 \]

Equivalent Mixed and Behavioral Strategies

**Definition**

We say that \(\sigma_i \in \Sigma_i\) and \(\beta_i \in \mathcal{B}_i\) are outcome equivalent if for all pure strategy profiles of the other players they lead to the same probability distribution over the terminal histories, i.e., for any \(s_{-i} \in S_{-i}\)
\[ p(\sigma, s_{-i})(z) = p(\beta, s_{-i})(z) \quad \text{for all } z \in Z \]

- In this example the mixed strategy and the behavioral strategy profiles induce the same outcome.
Equivalent Mixed and Behavioral Strategies

Let \( p \in (0, 1) \) and suppose the behavioral strategy is

\[ \beta(L|\{a_0, (a_0, R)\}) = p \]

\[ P^\beta(L) = \beta_1(L|a_0) = p \]
\[ P^\beta(R, L) = \beta_1(R|a_0, \beta_2(L|a_0, R)) = (1 - p)p \]
\[ P^\beta(R, R) = \beta_1(R|a_0, \beta_2(R|a_0, R)) = (1 - p)^2 \]

- What is the outcome equivalent mixed strategy?
- Any mixed strategy is a probability distribution over \( S_1 = \{L, R\} \)
- Only \( L \) and \((R, R)\) can receive positive probability
- There is no outcome equivalent mixed strategy

Perfect Recall

A game has perfect recall if no player forgets what she did or what she knew. More formally

**Definition (Perfect Recall)**

Suppose \( \Gamma \) is an extensive form game, \( x \in H \) and \( a \in A(x) \) such that \( y = (x, a, h) \in H \) for some sequence \( h \) and \( \iota(x) = \iota(y) \). \( \Gamma \) has perfect recall if \( w \in I(y) \) implies that there exists \( z \in I(x) \) such that \( w = (z, a, h') \) for some sequence \( h' \).

Kuhn proved that in games with perfect recall there is an equivalent behavioral strategy for every mixed strategy and conversely.

**Theorem (Kuhn 1953)**

In every finite extensive form game with perfect recall there is an outcome equivalent behavioral strategy for every mixed strategy and an outcome equivalent mixed strategy for every behavioral strategy.

Therefore, in extensive form games with perfect recall we are free to work with either mixed strategies or behavioral strategies; nothing is lost or added with this choice.

For example existence issue is more easily dealt with using mixed strategies whereas calculating equilibria is easier with behavioral strategies.
Equivalent Mixed and Behavioral Strategies

Given a behavioral strategy $\beta_i$ we can define an outcome equivalent mixed strategy $\sigma_i$ as follows. For any $s_i \in S_i$ let

$$\sigma_i(s_i) = \prod_{I \in I_i} \beta_i(s_i(I)|I)$$

There may be other outcome equivalent mixed strategies. Above definition gives just one.

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Equivalent Mixed and Behavioral Strategies

In the other direction, suppose we have a mixed strategy $\sigma_i$ and we want to derive an outcome equivalent behavioral strategy $\beta_i$.

**Definition**

A pure strategy $s_i$ is consistent with an information set $I$ if there exists a history $h \in I$ with which $s_i$ is consistent.

- $I_1^s$: set of information sets of $i$ with which $s_i$ is consistent
- $I_0^s$: set of information sets of $i$ with which some pure strategy in the support of $\sigma_i$ is consistent.
- For any $I \in I_i$ and $a \in A(I)$ define the set of all pure strategies that specify $a$ at information set $I$ as follows

$$S_i(a) = \{s_i \in S_i : s_i(I) = a\}$$

and those that are consistent with $I$ as

$$S_i(I) = \{s_i \in S_i : I \in I_i^s\}$$

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Suppose $\sigma_1(NH) = 1/2, \sigma_1(MH) = 1/2$

- What is the equivalent behavioral strategy?
- $S_1 = \{NH, NT, MH, MT\}$
- $\text{supp}(\sigma_1) = \{NH, MH\}$
- $I_1^{\sigma_1} = \{(a_0), \{(M, H), (M, T)\}\}$

Since $\{a_0\} \in I_1^{\sigma_1}$,

$$\beta_1(N|\{a_0\}) = \frac{\sum_{s_i \in S_i(\{a_0\}) \cap S_i(N)} \sigma_1(s_i)}{\sum_{s_i \in S_i(\{a_0\})} \sigma_1(s_i)}$$

$$S_1(\{a_0\}) = \{NH, NT, MH, MT\}, \quad S_1(N) = \{NH, NT\}$$

$$\beta_1(N|\{a_0\}) = \frac{\sigma_1(NH) + \sigma_1(NT)}{\sigma_1(NH) + \sigma_1(NT) + \sigma_1(MH) + \sigma_1(MT)} = \frac{1}{2}$$
\[ \sigma_1(NH) = 1/2, \sigma_1(MH) = 1/2 \]

- \{\{M, H\}, \{M, T\}\} \in \mathcal{I}^0_1
- \mathcal{S}_1(\{\{M, H\}, \{M, T\}\}) = \{MH, MT\}
- \mathcal{S}_1(H) = \{NH, MH\}

\[ \beta_1(H|\{\{M, H\}, \{M, T\}\}) = \frac{\sigma_1(MH)}{\sigma_1(MH) + \sigma_1(MT)} = 1 \]

Therefore, the equivalent behavioral strategy is
\[ \beta_1(N|\{a_0\}) = 1/2, \beta_1(H|\{\{M, H\}, \{M, T\}\}) = 1 \]

**Extensive Form Game Equilibria**

**Definition**

A **Nash equilibrium in mixed strategies** of an extensive form game \( \Gamma = (N, H, T, \beta, I, (u_i)) \) is a mixed strategy profile \( \sigma^* \in \Sigma \) such that for every player \( i \in N \)

\[ U_i(\sigma^*_i, \sigma^*_{-i}) \geq U_i(\sigma_i, \sigma^*_{-i}) \quad \text{for all } \sigma_i \in \Sigma_i \]

- A mixed strategy profile \( \sigma^* \) is a Nash equilibrium of an extensive form game \( \Gamma' \) if and only if it is a Nash equilibrium of its strategic form \( G_{\Gamma'} = (N, (S_i), (U_i)) \).
- A mixed strategy profile \( \sigma^* \) is a Nash equilibrium of an extensive form game \( \Gamma' \) if and only if no player has a profitable pure strategy deviation.
- Every finite extensive form game has a mixed strategy Nash equilibrium.

The equivalent mixed strategy to \( \beta_1(N|\{a_0\}) = 1/2, \beta_1(H) = 1 \) is given by

\[ \sigma_1(NH) = \beta_1(N|\{a_0\})\beta_1(H|\{\{M, H\}, \{M, T\}\}) = 1/2 \]
\[ \sigma_1(NT) = \beta_1(N|\{a_0\})\beta_1(T|\{\{M, H\}, \{M, T\}\}) = 0 \]
\[ \sigma_1(MH) = \beta_1(M|\{a_0\})\beta_1(H|\{\{M, H\}, \{M, T\}\}) = 1/2 \]
\[ \sigma_1(MT) = \beta_1(M|\{a_0\})\beta_1(T|\{\{M, H\}, \{M, T\}\}) = 0 \]

**Extensive Form Game Equilibria**

**Definition**

A **Nash equilibrium in behavioral strategies** of an extensive form game \( \Gamma = (N, H, T, \beta, I, (u_i)) \) is a behavioral strategy profile \( \beta^* \in \mathcal{B} \) such that for every player \( i \in N \)

\[ U_i(\beta^*_i, \beta^*_{-i}) \geq U_i(\beta_i, \beta^*_{-i}) \quad \text{for all } \beta_i \in \mathcal{B}_i \]

In extensive form games with perfect recall there is an equivalent behavioral strategy for every mixed strategy and conversely. This implies the following.

- A behavioral strategy profile \( \beta^* \) is a Nash equilibrium of an extensive form game \( \Gamma' \) with perfect recall if and only if no player has a profitable pure strategy deviation.
- Every finite extensive form game with perfect recall has a behavioral strategy Nash equilibrium.
Extensive Form Game Equilibria

**Definition**
Let $\Gamma = (N, H, \iota, \beta, (u_i))$ be an extensive form game. A mixed strategy profile $\sigma^* \in \Sigma$ is a subgame perfect equilibrium in mixed strategies of $\Gamma$ if $\sigma^*$ induces a Nash equilibrium in every subgame of $\Gamma$.

**Definition**
Let $\Gamma = (N, H, \iota, \beta, (u_i))$ be an extensive form game. A behavioral strategy profile $\beta^* \in B$ is a subgame perfect equilibrium in behavioral strategies of $\Gamma$ if $\beta^*$ induces a Nash equilibrium in every subgame of $\Gamma$.

- Every finite extensive form game has a mixed strategy subgame perfect equilibrium.
- Every finite extensive form game with perfect recall has a behavioral strategy subgame perfect equilibrium.

### Extensive Form Game Equilibria: Example

- The unique Nash equilibrium is also the unique SPE.
- We could have applied the backward induction like algorithm to calculate SPE.
  - Unique Nash equilibrium of the 'smallest' subgame is $\sigma_1(H) = \sigma_2(H) = 1/2$ with payoff vector $(0, 0)$.
  - Given that, the unique Nash equilibrium of the reduced game is $M$

![Extensive Form Game Equilibria: Example](image)

### Extensive Form Game Equilibria: Example

We can also calculate the SPE in behavioral strategies.
- In the subgame that follows $M$ the unique NE in behavioral strategies is $\beta_1(H|\{(M, H), (M, T)\}) = 1/2, \beta_2(H|\{M\}) = 1/2$
- Given that we must have $\beta_1(M|\{a_0\}) = 1$

Unique SPE in behavioral strategies is $\beta_1(M|\{a_0\}) = 1, \beta_1(H|\{(M, H), (M, T)\}) = 1/2, \beta_2(H|\{M\}) = 1/2$

- It is also the unique Nash equilibrium in behavioral strategies
- Not surprisingly, it is outcome equivalent to $\sigma_1(MH) = \sigma_1(MT) = 1/2, \sigma_2(H) = 1/2$
An Application: Strategic Delegation in Cournot

The game has the following structure:

- Firm 1 offers a contract to her manager \( w : \mathbb{R}^2 \to \mathbb{R}_+ \), that specifies a monetary payment to her as a function of market outcome
- Manager accepts \((y)\) or rejects \((n)\) the offer
- We assume that Firm 2 observes the contract and the manager’s decision
- If the manager rejects the offer, Firm 1 and Firm 2 play the standard Cournot game and manager receives his outside option \( w_m = 0 \)
- If the manager accepts the offer, manager and Firm 2 simultaneously choose output levels \( q_1, q_2 \geq 0 \)

The profit function of firm \( i \) is given by

\[
\pi_i(q_1, q_2) = \begin{cases} 
(a - c - b(q_1 + q_2))q_i, & q_1 + q_2 \leq a/b \\
-cq_i, & q_1 + q_2 > a/b 
\end{cases}
\]

We assume \( 5c > a > 2c \geq 0 \) and \( b > 0 \)

Payoff functions are given by

\[
u_1(w, \theta, q_1, q_2) = \begin{cases} 
\pi_1(q_1, q_2) - w(q_1, q_2), & \text{if } \theta = y \\
\pi_1(q_1, q_2), & \text{if } \theta = n 
\end{cases}
\]

\[
u_m(w, \theta, q_1, q_2) = \begin{cases} 
w(q_1, q_2), & \text{if } \theta = y \\
0, & \text{if } \theta = n 
\end{cases}
\]

\[
u_2(w, \theta, q_1, q_2) = \pi_2(q_1, q_2)
\]

Denote the subgame that follows history \((w, y)\) by \( \Gamma_{wy} \) and the one that follows \((w, n)\) by \( \Gamma_{wn} \)

\[
u_m((\alpha, \beta), y, q_1, q_2) = \beta + (1 - \alpha)\pi_1(q_1, q_2) + \alpha R_1(q_1, q_2)
\]

where \( \alpha \in [0, 1] \)

Therefore, a contract can be completely characterized by \((\alpha, \beta)\).

So, the manager’s payoff function after accepting a contract is

\[
u_m((\alpha, \beta), y, q_1, q_2) = \begin{cases} 
\beta + R_1(q_1, q_2) - (1 - \alpha)cq_1, & q_1 + q_2 \leq a/b \\
\beta - (1 - \alpha)cq_1, & q_1 + q_2 > a/b 
\end{cases}
\]

except for the constant \( \beta \), this is equivalent to the profit function of firm 1 with cost parameter \((1 - \alpha)c\)
An Application: Strategic Delegation in Cournot

NE of the subgame $\Gamma_{(\alpha, \beta)y}$

- $\beta$ has no effect on optimal output level for the manager
- Therefore, he acts like a firm 1 with cost parameter $(1 - \alpha)c$
- So, the best response correspondences are

$$B_m(q_2) = \begin{cases} \frac{a - (1 - \alpha)c - bq_2}{2b}, & q_2 < \frac{a - (1 - \alpha)c}{b} \\ 0, & q_2 \geq \frac{a - (1 - \alpha)c}{b} \end{cases}$$

$$B_2(q_1) = \begin{cases} \frac{a - c - bq_1}{2b}, & q_1 < \frac{a - c}{b} \\ 0, & q_1 \geq \frac{a - c}{b} \end{cases}$$

- Therefore, we have

$$q_1((\alpha, \beta), y) = \frac{a - c + 2\alpha c}{3b}, \quad q_2((\alpha, \beta), y) = \frac{a - c - \alpha c}{3b}$$

An Application: Strategic Delegation in Cournot

Firm 1’s contract decision

- The highest payoff that Firm 1 can get by making an offer that the manager accepts is found by solving the following problem

$$\max_{(\alpha, \beta)} \{ \pi_1((\alpha, \beta), y) - u_m((\alpha, \beta), y) \} \quad \text{s.t.} \quad u_m((\alpha, \beta), y) \geq 0$$

- At the solution the constraint must hold with equality. Substituting the constraint and the expression for $\pi_1((\alpha, \beta), y)$ in the objective function the problem becomes

$$\max_{(\alpha, \beta)} \frac{(a - c - \alpha c)(a - c + 2\alpha c)}{9b}$$

- Objective function does not depend on $\beta$. If optimal $\alpha$ is in the interior of $[0, 1]$, then the first order condition must hold with equality

$$ac - c^2 - 4c^2\alpha = 0$$

which is solved as $\alpha = \frac{a - c}{4c}$, which is indeed in the interior of $[0, 1]$. 

An Application: Strategic Delegation in Cournot

NE payoffs in the subgame $\Gamma_{(\alpha, \beta)y}$

- $u_m((\alpha, \beta), y) = \frac{(a - c + 2\alpha c)^2}{9b} + \beta$
- $u_1((\alpha, \beta), y) = \pi_1((\alpha, \beta), y) - u_m((\alpha, \beta), y)$
- $u_2((\alpha, \beta), y) = \frac{(a - c - \alpha c)^2}{9b}$

where

$$\pi_1((\alpha, \beta), y) = \frac{(a - c - \alpha c)(a - c + 2\alpha c)}{9b}$$

- The manager will accept (reject) if

$$u_m((\alpha, \beta), y) > (<)u_m((\alpha, \beta), y) = 0$$

and is indifferent in case of equality.

An Application: Strategic Delegation in Cournot

Therefore, Firm 1 can get arbitrarily close to the following payoff by offering a contract that the manager accepts

$$\frac{(a - c)^2}{8b}$$

- The payoff she gets by a contract that the manager rejects is

$$\frac{(a - c)^2}{9b}$$

- This implies that in any SPE the manager must be accepting the equilibrium contract
Since in any SPE in which the manager accepts $\alpha = (a - c)/4c$, the following must hold in any SPE:

$$u_m((\alpha, \beta), y) = \frac{(a - c)^2}{4b} + \beta \geq 0$$

Is it possible that

$$\frac{(a - c)^2}{4b} + \beta > 0$$

No, because Firm 1 could decrease $\beta$ slightly, still get the manager to accept, and increase payoff. Therefore, in any SPE

$$\beta = -\frac{(a - c)^2}{4b}$$

In the unique SPE

$$q_1 = \frac{a - c}{2b}, \ q_2 = \frac{a - c}{4b}$$

These are exactly the Stackelberg output levels when Firm 1 is the leader.

Firm 1 uses delegation strategically, i.e., as a commitment device.