Subgame Perfect Equilibrium

- Strategic form of the game

\[
\begin{array}{c|cc}
& L & R \\
O & 1,3 & 1,3 \\
T & 2,1 & 0,0 \\
B & 0,2 & 0,1 \\
\end{array}
\]

- Set of Nash equilibria

\[ N(\Gamma) = \{(T, L), (O, R)\} \]

- What is the set of SPE?

\[ \text{SPE}(\Gamma) = \{(T, L), (O, R)\} \]

- \((O, R)\) equilibrium is not plausible: \(R\) is strictly dominated for player 2
- SPE does not test for sequential rationality at every non-singleton information set
Sequential Rationality

- In this game sequential rationality of player 2 requires playing \( L \).
- How about in the following game?

```
                  O
                 / \
                /   \ 1
               /     \ B
              /       \
             /         \
            T         R
               |     |
              2       1
```

- Depends on player 2’s beliefs.
- We have to introduce beliefs in addition to strategies.
- ... and require sequential rationality at every information set given beliefs.
- This immediately eliminates \((O, R)\) equilibrium for the game in the previous slide.

Beliefs and Assessments

Unless noted otherwise, from now on we restrict our analysis to extensive form games
- with perfect recall, and
- in which every information set is finite.

**Definition (Belief System)**

A belief system is a collection of probability distributions \( \mu = (\mu(.|I))_{I \in I} \), where \( \mu(h|I) \) denotes the probability assigned to \( h \in I \). We denote the set of all possible beliefs by \( \mathcal{M} \).

**Definition (Assessment)**

An assessment \((\beta, \mu) \in \mathcal{B} \times \mathcal{M}\) is a behavioral strategy profile combined with a belief system.
Outcomes and Expected Payoffs

- Take any assessment \((\beta, \mu) \in \mathcal{B} \times \mathcal{M}\)
- Conditional probability assigned to history \(h\) given that history \(h'\) has been reached: \(P^\beta(h|h')\)
  - If there is no sequence \((a_1, a_2, \ldots, a_k)\) such that \(h = (h', a_1, a_2, \ldots, a_k)\), then \(P^\beta(h|h') = 0\)
  - If there is a sequence \((a_1, a_2, \ldots, a_k)\) such that \(h = (h', a_1, a_2, \ldots, a_k)\), then let \(h_0 = h', h_l = (h', a_1, a_2, \ldots, a_l)\), for any \(l = 1, 2, \ldots, k - 1\), and define
    \[
    P^\beta(h|h') = \prod_{l=0}^{k-1} \beta_{l(h_l)}(a_{l+1}|h_l)
    \]
- Perfect recall implies that each \(h_l\) lies in a different information set and hence events \(\{h_{l+1} \text{ occurs conditional on } h_l \text{ occurring}\}\) are independent and multiplying their probabilities makes sense.

Outcomes and Expected Payoffs

- Conditional probability assigned to history \(h\) given that information set \(I\) has been reached: \(P^{\beta, \mu}(h|I)\)
  \[
  P^{\beta, \mu}(h|I) = \sum_{h' \in I} \mu(h'|I) P^\beta(h|h')
  \]
  - Note that there is at most one \(h' \in I\) such that \(P^\beta(h|h') > 0\).
- Expected payoff of the assessment \((\beta, \mu)\) conditional on reaching information set \(I\) to the player who is moving at that information set
  \[
  U_{t(I)}(\beta, \mu|I) = \sum_{z \in \mathcal{Z}} P^{\beta, \mu}(z|I) u_t(z)
  \]
Sequential Rationality

**Definition (Sequential Rationality)**

An assessment \((\beta, \mu) \in \mathcal{B} \times \mathcal{M}\) is **sequentially rational** if for all \(I \in \mathcal{I}\)

\[
\beta \iota(I) \in \argmax_{\beta' \iota(I) \in \mathcal{B}_\iota(I)} U_{\iota(I)} \left( (\beta'_\iota(I), \beta_{-\iota(I)}), \mu | I \right)
\]

---

**Is Sequential Rationality Sufficient?**

Consider the following game:

![Game Tree Diagram]

- The following assessment is sequentially rational
  \[
  \beta_1(L|\{a_0\}) = 1, \beta_2(r|\{L, R\}) = 1, \mu(R|\{L, R\}) = 1
  \]

- Yet, the strategy profile is not even a Nash equilibrium of this game
- There is something unsatisfactory about the beliefs
- We would like beliefs to be consistent with strategies
Weak Sequential Equilibrium
For any $\beta \in \mathcal{B}$ and $h \in H$ let $I(h)$ be the information set that contains $h$, and

$$P^\beta(I(h)) = \sum_{h' \in I(h)} P^\beta(h')$$

Definition (Weak Consistency)
An assessment $(\beta, \mu) \in \mathcal{B} \times \mathcal{M}$ is weakly consistent if the belief system is derived from the strategies using Bayes’ rule whenever possible. In other words, for every $h \in H$ such that $P^\beta(I(h)) \neq 0$

$$\mu(h|I) = \frac{P^\beta(h)}{P^\beta(I(h))}$$

Definition (Weak Sequential Equilibrium)
An assessment $(\beta, \mu) \in \mathcal{B} \times \mathcal{M}$ is a weak sequential equilibrium if it satisfies sequential rationality and weak consistency.

Is Weak Sequential Equilibrium Good Enough?
Consider the following game

- The following assessment is a weak sequential equilibrium
  $$\beta_1(O|\{a_0\}) = 1, \beta_1(A|\{E\}) = 1, \beta_2(F|\{(E, F), (E, A)\}) = 1$$
  $$\mu((E, F)|\{(E, F), (E, A)\}) = 1$$
- Weak consistency is trivially satisfied since player 2’s information set is off-the-equilibrium
- However, the strategy profile is not a SPE
Consistency

- If a behavioral strategy profile $\beta$ assigns a strictly positive probability to every action, we say $\beta$ is completely mixed and write $\beta \gg 0$.
- Clearly, if $\beta \gg 0$, then $P^{\beta}(I(h)) > 0$ for all $h \in H$.
- For every $\beta \gg 0$, there is a unique belief system $\mu$ such that $(\beta, \mu)$ is weakly consistent.

**Definition (Consistency)**

An assessment $(\beta, \mu) \in \mathcal{B} \times \mathcal{M}$ is consistent if there exists a sequence $(\beta^n, \mu^n) \to (\beta, \mu)$ such that $\beta^n \gg 0$ and $(\beta^n, \mu^n)$ is weakly consistent for all $n$.

- The intuition is that probability of an event conditional on zero probability events must approximate the probability that is derived from “nearby” completely mixed strategies.
- For on-the-path information sets, weak consistency uniquely determines beliefs and implies consistency.

Sequential Equilibrium

**Definition (Sequential Equilibrium)**

An assessment $(\beta, \mu) \in \mathcal{B} \times \mathcal{M}$ is a sequential equilibrium if it satisfies sequential rationality and consistency.
Sequential Equilibrium

Consider again

\[ \begin{align*}
\beta_1(O|\{a_0\}) &= 1, \\
\beta_1(A|\{E\}) &= 1, \\
\beta_2(F|\{(E, F), (E, A)\}) &= 1 \\
\mu((E, F)|\{(E, F), (E, A)\}) &= 1
\end{align*} \]

and the weak sequential equilibrium

\[ \begin{align*}
\beta_1(O|\{a_0\}) &= 1, \\
\beta_1(A|\{E\}) &= 1, \\
\beta_2(F|\{(E, F), (E, A)\}) &= 1 \\
\mu((E, F)|\{(E, F), (E, A)\}) &= 1
\end{align*} \]

For any weakly consistent \((\beta^n, \mu^n)\) we must have

\[ \mu^n((E, F)|\{(E, F), (E, A)\}) = \beta^n_1(F|\{E\}) \]

The convergence condition requires that \(\beta^n_1(F|\{E\}) \to 0\) and hence

\[ \mu^n((E, F)|\{(E, F), (E, A)\}) \to 0, \]

But \(\mu((E, F)|\{(E, F), (E, A)\}) = 1\) in the assessment.

Sequential Equilibrium

What is the set of sequential equilibria for this game?

There are three possibilities

1. \(\beta_2(F|\{(E, F), (E, A)\}) = 1\)
   - sequential rationality implies \(\beta_1(A|\{E\}) = 1\)
   - consistency implies \(\mu((E, A)|\{(E, F), (E, A)\}) = 1\)
   - sequential rationality implies \(\beta_2(F|\{(E, F), (E, A)\}) = 0\), contradiction
Sequential Equilibrium

\[ \beta_2(F|(E, F), (E, A)) = 0 \]
- sequential rationality implies \( \beta_1(A|E) = 1 \)
- consistency implies \( \mu((E, A)|(E, F), (E, A)) = 1 \)
- sequential rationality implies \( \beta_2(F|(E, F), (E, A)) = 0 \)
- sequential rationality implies \( \beta_1(E|a_0) = 1 \)

\( (\beta^n, \mu^n) \) below satisfies weak consistency and convergence criteria

\[
\beta^n_1(E|a_0) = \beta^n_1(A|E) = \beta^n_2(A|(E, F), (E, A)) = 1 - \frac{1}{n + 1}
\]

\[
\mu^n((E, A)|(E, F), (E, A)) = 1 - \frac{1}{n + 1}
\]

Sequential Equilibrium

\[ \beta_2(F|(E, F), (E, A)) \in (0, 1) \]
- sequential rationality of player 1 implies \( \beta_1(A|E) = 1 \)
- consistency implies \( \mu((E, A)|(E, F), (E, A)) = 1 \)
- sequential rationality of player 2 implies \( \beta_2(A|(E, F), (E, A)) = 1 \), a contradiction
Let $\mu = \mu(D|\{D, (C, d)\})$, $\alpha = \beta_1(D|\{a_0\})$, $\beta = \beta_2(d|\{C\})$, $\gamma = \beta_3(L|\{D, (C, d)\})$

There are three possible types of sequential equilibria:

1. $\gamma = 1$
2. $\gamma = 0$
3. $\gamma \in (0, 1)$

Let’s investigate each.

$\gamma = 1$

- Sequential rationality (SR) of player 2 implies $\beta = 1$
- SR of player 1 implies $\alpha = 0$
- Bayes' rule (BR) implies $\mu = 0$
- But then SR of player 3 implies $\gamma = 0$, a contradiction
Selten’s Horse

\[ \gamma = 0 \]

- Sequential rationality (SR) of player 2 implies \( \beta = 0 \)
- SR of player 1 implies \( \alpha = 0 \)
- BR has no bite, but SR of player 3 implies \( 1 - \mu \geq 2\mu \) or \( \mu \leq 1/3 \)

Selten’s Horse

For any \( \mu \in (0, 1/3] \), let

\[ \alpha^n = \frac{\mu}{n+1}, \beta^n = \frac{1 - \mu}{n+1}, \gamma^n = \frac{1}{n+1}, \mu^n = \frac{1}{\frac{1}{\mu} - \frac{1-\mu}{n+1}} \]

For \( \mu = 0 \), let

\[ \alpha^n = \frac{1}{(n+1)^2}, \beta^n = \frac{1}{n+1}, \gamma^n = \frac{1}{n+1}, \mu^n = \frac{n+1}{n + (n+1)^2} \]

Note that \( \alpha^n \to \alpha, \beta^n \to \beta, \gamma^n \to \gamma, \mu^n \to \mu \) and

\[ \mu^n = \frac{\alpha^n}{\alpha^n + (1 - \alpha^n)\beta^n} \]

so that consistency is satisfied. Therefore, the following is a class of SE whose members differ only in off-the-equilibrium beliefs.

\[ \{(\alpha, \beta, \gamma, \mu) : \alpha = \beta = \gamma = 0, \mu \leq 1/3\} \]
Selten’s Horse

\[ \gamma \in (0, 1) \]
Sequential rationality (SR) of player 3 implies \( \mu = 1/3 \)
Let's consider the three possibilities:

1. \( \beta = 1 \): SR of 1 implies \( \alpha = 0 \). But then BR implies \( \mu = 0 \), a contradiction.

2. \( \beta \in (0, 1) \): SR of 2 implies \( \gamma = 1/4 \), which, by SR of 1, implies \( \alpha = 0 \). But then BR implies \( \mu = 0 \), a contradiction.

3. \( \beta = 0 \): SR of 2 implies \( \gamma \leq 1/4 \), which, by SR of 1, implies \( \alpha = 0 \).

Selten’s Horse

So let \( \mu = 1/3, \alpha = 0, \beta = 0, \gamma \in (0, 1/4] \) and

\[
\alpha^n = \frac{1}{3(n+1)}, \beta^n = \frac{2}{3(n+1)}, \gamma^n = \frac{1}{n+1}, \mu^n = \frac{1}{3 - \frac{2}{3(n+1)}}
\]

Note that \( \alpha^n \to \alpha, \beta^n \to \beta, \gamma^n \to \gamma, \mu^n \to 1/3 \) and

\[
\mu^n = \frac{\alpha^n}{\alpha^n + (1 - \alpha^n)\beta^n}
\]

so that consistency is satisfied. Combining this with the class of SE found previously, we have

**Proposition**

*The set of SE behavioral strategy profile is given by*

\[
\{ (\alpha, \beta, \gamma, \mu) : \alpha = \beta = 0, \gamma \in [0, 1/4]\}
\]

*Each such strategy profile can be combined with some \( \mu \leq 1/3 \) to constitute a SE.*
Properties of Sequential Equilibrium

Proposition (Kreps and Wilson (82))
Every finite extensive form game with perfect recall has a sequential equilibrium.

Proposition (Kreps and Wilson (82))
Every sequential equilibrium is a subgame perfect equilibrium.

Proposition (One-Deviation Property)
A consistent assessment \((\beta, \mu)\) is sequentially rational if, and only if, for all \(I \in \mathcal{I}\) and all \(\beta' \in \mathcal{B}\) satisfying \(\beta'_i(I) = \beta_i(I)\) for all \(I' \neq I\), the following is true
\[
U_i(I) \geq U_i(I, \beta', \mu|I) \geq U_i(I, \beta, \mu|I)
\]

Proof.
See Hendon et al (1996), Games and Economic Behavior
Perfect Bayesian Equilibrium

- Consistency is difficult to work with which makes sequential equilibrium difficult to use in applications
- A widely used alternative: perfect Bayesian equilibrium
  - Developed by Fudenberg and Tirole (1991) for a particular class of extensive form games
  - Extensive form games with incomplete information and observed actions
  - Puts more restrictions on off-the-equilibrium beliefs than weak sequential equilibrium but less restrictions than sequential equilibrium

Games with Observed Actions and Perfect Bayesian Equilibrium

In an extensive form game with incomplete information and observed actions

- Each player has a type that is revealed only to the player at the beginning of the game
- After each history in the game, players simultaneously choose actions, which are revealed before players choose actions again

Perfect Bayesian equilibrium requires that

- Posterior beliefs about a player’s type is derived from prior beliefs using Bayes’ rule even at off-the-equilibrium information sets, as long as the player’s action has positive probability conditional on the history
- Beliefs about a player’s type depend only on that player’s actions
- Beliefs are such that the players’ types are independently distributed
Games with Observed Actions

An extensive form game with incomplete information and observed actions is composed of the following elements:

1. \( N \): a finite set of players
2. \( H \): a set of histories
3. \( \iota : H \setminus Z \to 2^N \): the player function
   - Players in \( \iota(h) \) move simultaneously after history \( h \)
   - There is a collection of sets \( \{ A_i(h) \}_{i \in \iota(h)} \) such that
     \[
     A(h) = \{ (h, a) \in H \} = \times_{i \in \iota(h)} A_i(h)
     \]
   - Player \( i \) chooses from \( A_i(h) \) after history \( h \)

4. \( \Theta_i \): a finite set of types for player \( i \)
   - \( \Theta = \times_{i \in N} \Theta_i \): set of type profiles
5. \( p \): Nature’s probability distribution over \( \Theta \)
   - Nature moves only once at the beginning of the game choosing a type profile \( \theta \) according to \( p \)
   - We assume types are independent: \( p(\theta) = \prod_{i \in N} p_i(\theta_i) \)
     - \( p_i \) is the marginal distribution over \( \Theta_i \)
     - Assume \( p_i(\theta_i) > 0, \forall \theta_i \in \Theta_i \)
6. An information partition \( I \)
   - Each player observes only her own type and the history so far
   - Player \( i \)’s information set after history \( h \) is given by
     \[
     I(\theta_i, h) = \{ ((\theta_i, \theta'_{-i}), h) : \theta'_{-i} \in \Theta_{-i} \}
     \]
7. For each \( i \in N \), a von Neumann-Morgenstern payoff function
   \[
   u_i : \Theta \times Z \to \mathbb{R}
   \]
Games with Observed Actions

An assessment is given by a behavioral strategy profile and belief system $(\beta, \mu)$, where

- $\beta_i(a_i|\theta_i, h)$ denotes the probability with which player $i$ of type $\theta_i$ chooses action $a_i$ after history $h$
- $\mu(\theta_i|h)$ denotes the (common) probability assigned, after history $h$, to player $i$ being of type $\theta_i$

**Definition (Perfect Bayesian Equilibrium)**

An assessment $(\beta, \mu) \in B \times M$ is a perfect Bayesian equilibrium (PBE) if it is sequentially rational and for all $h \in H \setminus Z$

1. **Strong Bayes’ rule:** For all $i \in \iota(h)$ and $a_i \in A_i(h)$, if there exists $\theta_i' \in \Theta_i$ with $\mu(\theta_i'|h) > 0$ and $\beta_i(a_i|\theta_i', h) > 0$ (i.e., $a_i$ has positive probability conditional on $h$), then for any $\theta_i \in \Theta_i$

$$\mu(\theta_i|h, a) = \frac{\mu(\theta_i|h)\beta_i(a_i|\theta_i, h)}{\sum_{\tilde{\theta}_i \in \Theta_i} \mu(\tilde{\theta}_i|h)\beta_i(a_i|\tilde{\theta}_i, h)}$$

2. **Independence:** For all $\theta \in \Theta$

$$\mu(\theta|h) = \prod_{i \in N} \mu(\theta_i|h)$$

3. **Action determined beliefs:** For all $\theta_i \in \Theta_i$
   - If $i \notin \iota(h)$ and $a \in A(h)$, then $\mu(\theta_i|h, a) = \mu(\theta_i|h)$
   - If $i \in \iota(h)$ and $a, a' \in A(h)$ and $a_i = a'_i$, then $\mu(\theta_i|h, a) = \mu(\theta_i|h, a')$
PBE Beliefs

- **Strong Bayes rule:**
  - The denominator is the probability assigned to player $i$ playing action $a_i$ conditional upon reaching history $h$, whereas the numerator is the probability assigned to player $i$ playing action $a_i$ and having type $\theta_i$ conditional upon reaching history $h$.
  - It applies even if
    - $h$ has zero probability under $\beta$,
    - $h$ has positive probability but, under $\beta$, $a$ is a zero probability event after $h$. This, of course, must be due to some other player deviating from $a$, as there is at least one type of player $i$ who put positive probability on $a_i$.
  - It says that once beliefs are formed about player $i$ at an off-the-path information set, they should, together with $\beta_i$, form the basis for updating at subsequent information sets until player $i$ deviates from $\beta_i$ again. In that sense it prevents belief reversals.

- **Action determined beliefs:** Other players’ actions cannot give information about a player’s type. It fits our assumption that players’ strategies are independent.

The following figure illustrates the application of the Bayes’ rule

The standard Bayes’ rule leaves $\mu, \mu_1, \mu_2$ free. But the stronger rule defined above says that

- $\mu$ is free
- $\mu_1 = \mu$
- $\mu_2$ is free
Perfect Bayesian Equilibrium

Proposition

Every SE is a PBE.

Proposition

If each player has at most two types, or the game has at most two stages, then the sets of PBE and SE coincide.

Proof.

See Fudenberg and Tirole (1991), Journal of Economic Theory

Proposition (One-Deviation Property)

An assessment $\beta, \mu$ that satisfies the strong Bayes rule is sequentially rational if, and only if, for all $I \in \mathcal{I}$ and all $\beta' \in \mathcal{B}$ satisfying $\beta'(a|I) = \beta_\iota(I)(a|I)$ whenever $a \notin A(I)$, the following is true:

$$U_\iota(I)(\beta, \mu|I) \geq U_\iota(I)(\beta', \mu|I)$$

Proof.

See Hendon et al (1996), Games and Economic Behavior

PBE vs. SE

Consider the following (portion of a) game

- Player 1 has three types $l, m, r$, player 2 has only one type
- the game has reached a history $(h)$ where player 2 gives probability 1 to type $r$
- Strategies of the players are indicated by arrows, and beliefs in angled parentheses

![Game Diagram]
Do these beliefs satisfy the conditions for PBE?

Since there is no $\theta_1$ such that $\mu(\theta_1|h) > 0$ and $\beta_1(L|\theta_1, h) > 0$ beliefs following $L$ are free.

The same is true for beliefs that follow $M$.

Is the assessment consistent? Does there exist completely mixed $(\beta^n, \mu^n) \rightarrow (\beta, \mu)$ that satisfy Bayes’ rule for every $n$?

Bayes’ rule implies

$$\mu^n(m|h, L) = \frac{\mu^n(m|h) \beta^n(L|m, h)}{\mu^n(m|h) \beta^n(L|m, h) + \mu^n(l|h) \beta^n(L|l, h) + \mu^n(r|h) \beta^n(L|r, h)}$$

Divide the numerator and denominator by $\mu^n(m|h)$

$$\mu^n(m|h, L) = \frac{\beta^n(L|m, h)}{\beta^n(L|m, h) + \frac{\mu^n(l|h)}{\mu^n(m|h)} \beta^n(L|l, h) + \frac{\mu^n(r|h)}{\mu^n(m|h)} \beta^n(L|r, h)}$$

Consistency and $\mu^n(m|h, L) \rightarrow 1, \beta^n(L|l, h) \rightarrow 1$ imply that $\frac{\mu^n(l|h)}{\mu^n(m|h)} \rightarrow 0$. Similarly, consistency and $\mu^n(l|h, M) \rightarrow 1, \beta^n(M|m, h) \rightarrow 1$ imply that $\frac{\mu^n(m|h)}{\mu^n(l|h)} \rightarrow 0$, a contradiction.
PBE vs. SE

- This is an example proving that not every PBE beliefs are consistent, and hence not every PBE is a SE.
- Fudenberg and Tirole (1991) extend the definition to relative probabilities and establish equivalence between PBE and SE when beliefs about relative probabilities satisfy the three conditions given in the definition of PBE.
- Kohlberg and Reny (1997) provide another characterization of consistency that does not use sequences.

An Application: Reputation in Chain-Store Game

- Consider the following simple entry game with $a > 1$ and $0 < b < 1$
- Player 1 is a potential entrant who may enter ($E$) or stay out ($O$)
- Player $M$ is the incumbent monopolist who may fight ($F$) or accommodate ($A$)
- The following is the game tree, where the first payoff number is that of the monopolist

Unique SPE is $(E, A)$
An Application: Reputation in Chain-Store Game

Now consider the $K$ market version: the same game is played in markets $1, 2, \ldots, K$ against different entrants in sequence

- First, market $K$ is played against entrant $K$, second against entrant $K - 1$, and so on.
- Monopolist’s payoff is the sum of his payoffs in $K$ markets.
- Entrant $k$ observes the history of market interaction up to and including market $k + 1$.
- What is the SPE?
- In the unique SPE every entrant enters and the monopolist accommodates in every market independent of the history.
- Couldn’t the monopolist build a reputation for being a “tough” competitor by fighting in earlier markets so as to deter entry later on?
- Formalization of this idea requires introducing incomplete information.

There are two possible types of the monopolist: Tough ($\tau$) and Weak ($\omega$).
- Tough monopolist prefers to fight in any single market.
- Weak has the preferences given before.
- Nature determines the type of the monopolist: Tough with probability $\varepsilon \in (0, 1)$, Weak with $1 - \varepsilon$.
- Assume that there is no integer $k$ such that $\varepsilon = b^k$.
- Monopolist knows his type but entrants don’t.
- Here is the $K = 1$ version.

![Game Tree Diagram]
\[ K = 1 \]

\[ N = \{1, M\} \]
\[ H = \{a_0, O, E, (E, F), (E, A)\} \]
\[ \iota(a_0) = 1, \iota(E) = M \]
\[ \Theta_M = \{\omega, \tau\}, \Theta_1 = \{x\} \]
\[ p(\tau) = \varepsilon \]
\[ \text{For all } h : \iota(h) = 1, \ I(x, h) = \{((x, \theta_M), h) : \theta_M \in \{\omega, \tau\}\} \]
\[ \text{For all } h : \iota(h) = M, \ I(\theta_M, h) = \{((\theta_M, x), h) : \theta_M \in \{\omega, \tau\}\} \]

**Perfect Bayesian Equilibrium**

- Sequential Rationality (SR) \( \Rightarrow \beta_M(F|\omega, E) = 0, \beta_M(F|\tau, E) = 1 \)
- Action Determined Beliefs (ADB) \( \Rightarrow \mu(\tau|a_0) = \varepsilon \)
- ADB \( \Rightarrow \mu(\tau|E) = \varepsilon \)
- Expected payoff to entering is
  \[ (1 - \varepsilon)b + \varepsilon(b - 1) = b - \varepsilon \]

Therefore, SR \( \Rightarrow \)

\[ \beta_1(E|a_0) = \begin{cases} 
0, & \varepsilon > b \\
1, & \varepsilon < b 
\end{cases} \]
Let’s find PBE for the two markets version: Market 2 is played first and Market 1 second. Also assume $b^2 < \varepsilon < b$

$k=1$
For any market 2 history $h$, SR $\Rightarrow \beta_M(F|\omega, h, E) = 0, \beta_M(F|\tau, h, E) = 1$

$$\beta_1(E|h) = \begin{cases} 0, & \mu(\tau|h) > b \\ \in [0,1], & \mu(\tau|h) = b \\ 1, & \mu(\tau|h) < b \end{cases}$$

$k=2$
Let’s organize our search for PBE according to the way tough monopolist plays. There are three possibilities:

1. $\beta_M(F|\tau, E) = 0$
2. $\beta_M(F|\tau, E) = 1$
3. $\beta_M(F|\tau, E) \in (0,1)$

We will investigate each in turn.
\[ \beta_M(F|\tau, E) = 0 \]
\[ \beta_M(F|\omega, E) = 1 \]
\[ a. \beta_M(F|\omega, E) = 1 \]
\[ \star \text{SBR and ADB } \Rightarrow \mu(\tau|E, F) = 0, \mu(\tau|E, A) = 1 \]
\[ \star U_M((F, \beta_{-M}), \mu|\omega, E) = -1 + 0 < 0 + a = U_M((A, \beta_{-M}), \mu|\omega, E), \text{ contradiction (\dagger)} \]
\[ b. \beta_M(F|\omega, E) = 0 \]
\[ \star \text{SBR and ADB } \Rightarrow \mu(\tau|E, F) \in [0, 1], \mu(\tau|E, A) = \varepsilon \]
\[ \star U_M((A, \beta_{-M}), \mu|\tau, E) = -1 + 0 < 0 \leq U_M((F, \beta_{-M}), \mu|\tau, E), \text{ \dagger} \]
\[ c. \beta_M(F|\omega, E) \in (0, 1) \]
\[ \star \text{SBR and ADB } \Rightarrow \mu(\tau|E, F) = 0 \text{ and} \]
\[ \mu(\tau|E, A) = \frac{\varepsilon \times 1}{\varepsilon \times 1 + (1 - \varepsilon) \times \beta_M(A|\omega, E)} > \varepsilon \]
\[ \star U_M((F, \beta_{-M}), \mu|\omega, E) = -1 + 0 < 0 \leq U_M((A, \beta_{-M}), \mu|\omega, E), \text{ \dagger} \]

\[ \beta_M(F|\tau, E) = 1 \]
\[ a. \beta_M(F|\omega, E) = 1 \]
\[ \star \text{SBR and ADB } \Rightarrow \mu(\tau|E, F) = \varepsilon, \mu(\tau|E, A) \in [0, 1] \]
\[ \star U_M((F, \beta_{-M}), \mu|\omega, E) = -1 + 0 < 0 \leq U_M((A, \beta_{-M}), \mu|\omega, E), \text{ \dagger} \]
\[ b. \beta_M(F|\omega, E) = 0 \]
\[ \star \text{SBR and ADB } \Rightarrow \mu(\tau|E, F) = 1, \mu(\tau|E, A) = 0 \]
\[ \star U_M((A, \beta_{-M}), \mu|\omega, E) = 0 + 0 < -1 + a = U_M((F, \beta_{-M}), \mu|\omega, E), \text{ \dagger} \]
\( \beta_M(F|\tau, E) = 1 \) (cont’d)

c. \( \beta_M(F|\omega, E) \in (0, 1) \)

\[ \mu(\tau|E, F) = \frac{\varepsilon \times 1}{\varepsilon \times 1 + (1 - \varepsilon) \times \beta_M(F|\omega, E)} > \varepsilon \]

\[ \mu(\tau|E, A) = 0 \] and

\[ \mu(\tau|E, F) = \varepsilon \times 1 + (1 - \varepsilon) \times \beta_M(F|\omega, E) > \varepsilon \]

\[ \beta_1(O|E, F) = 1/a \Rightarrow b = \mu(\tau|E, F) \Rightarrow \]

\[ \beta_M(F|\omega, E) = \frac{(1 - b)\varepsilon}{b(1 - \varepsilon)} \]

\[ \varepsilon(b - 1) + (1 - \varepsilon) \left( b - \frac{(1 - b)\varepsilon}{b(1 - \varepsilon)} \right) = b - \frac{\varepsilon}{b} \]

Therefore, since we assumed \( \varepsilon > b^2 \), \( \beta_2(E|a_0) = 0 \)

\( \beta_M(F|\tau, E) \in (0, 1) \) is left as an exercise.

Therefore, the following is a PBE of the two-market game

\[ \beta_2(O|a_0) = 1, \beta_M(F|\tau, E) = 1, \beta_M(F|\omega, E) = \frac{(1 - b)\varepsilon}{b(1 - \varepsilon)} \]

\[ \beta_M(F|\omega, h, E) = 0, \beta_M(F|\tau, h, E) = 1 \]

\[ \mu(\tau|h) = \begin{cases} 
\varepsilon, & h \in \{a_0, O\} \\
0, & h = (E, A) \\
b, & h = (E, F) 
\end{cases} \]

\[ \beta_1(O|h) = \begin{cases} 
0, & h \in \{O, (E, A)\} \\
1/a, & h = (E, F) 
\end{cases} \]

In this equilibrium, even the weak monopolist fights earlier with positive probability so as to deter entry later on.

In this sense we may say that the weak monopolist “builds” reputation for being tough.
Now consider the $K$-market version. We will show that an equilibrium with the following features exists:

- Tough monopolist always fights
- Weak monopolist fights in the earlier markets and starts accommodating with positive probability towards the end.
- Entrants stay out in the earlier markets

Let $k = K, K - 1, \ldots, 2, 1$ and $h_k \in \{O, (E, A), (E, F)\}$ stand for the market $k$ sequence of events.

Let $k(h)$ be the market number that corresponds to history $h$. For any $h : \iota(h) = M$

$$\beta_M(F|\tau, h) = 1$$

$$\beta_M(F|\omega, h) = \begin{cases} 0, & k(h) = 1 \\ 1, & k(h) > 1 \text{ and } \mu(\tau|h) \geq b^{k(h)-1} \\ \frac{(1-b^{k(h)-1})\mu(\tau|h)}{(1-\mu(\tau|h))b^{k(h)-1}}, & k(h) > 1 \text{ and } \mu(\tau|h) < b^{k(h)-1} \end{cases}$$

For any $h : \iota(h) = k, k = K, K - 1, \ldots, 1$

$$\beta_k(O|h) = \begin{cases} 1, & \mu(\tau|h) > b^k \\ \frac{1}{a}, & \mu(\tau|h) = b^k \\ 0, & \mu(\tau|h) < b^k \end{cases}$$

$\mu(\tau|a_0) = \varepsilon$ and beliefs at market $k - 1$ as a function of beliefs at market $k$ and $h_k$ are given by

$$\mu(\tau|h, h_k) = \begin{cases} \mu(\tau|h), & h_k = O \\ \max\{b^{k-1}, \mu(\tau|h)\}, & h_k = (E, F) \text{ and } \mu(\tau|h) > 0 \\ 0, & h_k = (E, A) \text{ or } \mu(\tau|h) = 0 \end{cases}$$
Let’s verify that this is a PBE.

**Beliefs**

- Action determined beliefs (ADB) implies $\mu(\tau|a_0) = \varepsilon$ and $\mu(\tau|h, O) = \mu(\tau|h)$.
- If $\mu(\tau|h) = 0$ we either have $\mu(\tau|h, h_k) = 0$ by strong Bayes rule (SBR) or beliefs are free.
- Since $\beta_M(A|\tau, h) = 0$, $h_k = (E, A)$ implies $\mu(\tau|h, h_k) = 0$, by SBR, or $\mu(\tau|h, h_k)$ is free.
- If $\mu(\tau|h) > 0$ and $h_k = (E, F)$, SBR implies
  \[
  \mu(\tau|h, E, F) = \frac{\mu(\tau|h) \beta_M(F|\tau, h, E)}{\mu(\tau|h) \beta_M(F|\tau, h, E) + (1 - \mu(\tau|h)) \beta_M(F|\omega, h, E)}
  \]
  - If $\mu(\tau|h) \geq b^{k-1}$, then $\beta_M(F|\omega, h, E) = 1$, and hence $\mu(\tau|h, E, F) = \mu(\tau|h)$.
  - If $\mu(\tau|h) < b^{k-1}$, then
    \[
    \beta_M(F|\omega, h, E) = \frac{(1 - b^{k-1}) \mu(\tau|h)}{(1 - \mu(\tau|h)) b^{k-1}}
    \]
    and hence $\mu(\tau|h, E, F) = b^{k-1}$.

Therefore, beliefs satisfy the three conditions for PBE.

**Entrant $k$**

- If $\mu(\tau|h) \geq b^{k-1}$, then $\beta_M(F|\theta, h) = 1$, $\theta = \tau, \omega$, so $\beta_k(O|h) = 1$ is optimal.
- If $b^k < \mu(\tau|h) < b^{k-1}$, then probability of fight in market $k$ is
  \[
  \mu(\tau|h) + (1 - \mu(\tau|h)) \left(\frac{1 - b^{k-1}) \mu(\tau|h)}{(1 - \mu(\tau|h)) b^{k-1}}\right) = \frac{\mu(\tau|h)}{b^{k-1}}
  \]
  and the expected payoff to entering is
  \[
  \frac{b^k - \mu(\tau|h)}{b^{k-1}} \begin{cases} 
  < 0, & \mu(\tau|h) > b^k \\
  = 0, & \mu(\tau|h) = b^k \\
  > 0, & \mu(\tau|h) < b^k
  \end{cases}
  \]
  and hence the entrant’s strategy is sequentially rational.
Tough monopolist
By accommodating the tough monopolist gets $-1$ today and $0$ afterwards, whereas by fighting he gets $0$ today and at worst $0$ in the future. Therefore, always fighting is optimal.

Weak monopolist
- If $k(h) = 1$, there is no tomorrow and $A$ is optimal. So, suppose $k(h) > 1$ and note that playing $A$ brings a total payoff of zero.
- If $\mu(\tau|h) > b^{k(h)-1}$, then $\mu(\tau|h, E, F) = \mu(\tau|h) > b^{k(h)-1}$ and therefore $\beta_{k-1}(O|h, E, F) = 1$, and the monopolist’s payoff is at least $-1 + a > 0$. In this case fighting is optimal.
- If $\mu(\tau|h) = b^{k(h)-1}$, then $\mu(\tau|h, E, F) = b^{k(h)-1}$ and therefore $\beta_{k-1}(O|h, E, F) = 1/a$. This implies that the payoff to $F$ is at least $-1 + \frac{1}{a}a = 0$, and hence fighting is again optimal.

Weak monopolist (cont’d)
Optimality in the case $\mu(\tau|h) < b^{k(h)-1}$ can be proved by induction.
- In market $2$, $\mu(\tau|h, E, F) = b$ and therefore $\beta_{k-1}(O|h, E, F) = 1/a$. The monopolist gets an expected payoff of $-1 + \frac{1}{a}a = 0$ by fighting. Therefore, mixing is optimal.
- Now suppose mixing is optimal at market $k$ when $\mu(\tau|h) < b^{k-1}$. This implies that, if $\mu(\tau|h) < b^{k-1}$, $A$ and $F$ bring the same expected total payoff (of zero) starting at $k$.
- Now suppose that $\mu(\tau|h) < b^k$ in market $k + 1$. If the monopolist fights in this market, $\mu(\tau|h, E, F) = b^k$ in market $k$. This implies that $\beta_k(O|h, E, F) = 1/a$. Furthermore, since $\mu(\tau|h, E, F) = b^k < b^{k-1}$, if entry occurs the monopolist gets a total payoff of zero starting from market $k$, by the induction hypothesis. Therefore, the payoff to fighting is $-1 + \frac{1}{a}a + (1 - \frac{1}{a}) \times 0 = 0$ in market $k + 1$. This proves that if $\mu(\tau|h) < b^k$, then mixing is optimal in market $k + 1$. 
Let \( \min \{ k : b^k < \varepsilon \} = 4 \). The following figure illustrates the evolution of beliefs and the equilibrium path.

The game starts in market \( K \) with belief \( \mu_K = \varepsilon \). Since \( \mu_K > b^K \), entrant stays out and the next period belief \( \mu_{K-1} = \varepsilon \). If the out-of-equilibrium outcome \( (E, F) \) \((E, A)\) were to occur \( \mu_{K-1} = \varepsilon \) \((\mu_{K-1} = 0)\). The game continues in this manner until market 4.

4 In market 4 we still have \( \mu_4 = \varepsilon > b^4 \), and therefore entrant stays out. However, since \( \mu_4 < b^4 \), weak monopolist would mix between \( E \) and \( F \) if entry were to occur.

3 In market 3 \( \mu_3 = \varepsilon < b^3 \), and entrant enters. Since \( \mu_3 < b^2 \), weak monopolist mixes between \( E \) and \( F \).

2 If market 3 outcome is \( (E, A) \), then \( \mu_2 = 0 \), and entrant enters in market 2. If the outcome in market 3 is \( (E, F) \), then \( \mu_2 = b^2 \) and entrant mixes. Since \( \mu_2 < b \), weak monopolist mixes as well.

1 In market 1, weak monopolist accommodates if entry occurs. Entrant enters if the outcome of market 2 is \( O \) or \( (E, A) \), and mixes if it is \( (E, F) \).

Reputation in Chain-Store Game

- Note that the entrants stay out until market \( \min \{ k : b^k < \varepsilon \} - 1 \), which is independent of the number of markets. Therefore, in the above equilibrium

\[
\lim_{N \to \infty} \frac{U_M(\beta, \mu)}{N} = a, \quad \text{for any } \varepsilon > 0
\]

On the other hand, if \( \varepsilon = 0 \), in the unique SE,

\[
\frac{U_M(\beta, \mu)}{N} = 0
\]

This shows that a small amount of uncertainty can lead to large changes in equilibrium outcomes.
Reputation in Chain-Store Game

- There are other Nash equilibria of this game that are not sequential equilibria. For example, “always fight” for both types of the monopolist and “always stay out” for the entrant is such an equilibrium.

- For some parameters, there are also other SE with different outcomes. But Kreps and Wilson (1982, Journal of Economic Theory) show that if \( \varepsilon \neq b^k \) for \( k \leq K \) and beliefs are such that

\[
\mu(\tau|h, F) \geq \mu(\tau|h, A)
\]

for any history \( h \), then the equilibrium we described is unique on the equilibrium path.