Admission Policies for a Two Class Loss System

E. Lerzan Örmeci
Department of Industrial Engineering
Koç University, Istanbul, Turkey

Apostolos Burnetas
Department of Operations
Weatherhead School of Management
Case Western Reserve University
Cleveland, OH 44106. USA

Jan van der Wal
Department of Mathematics
Technical University of Eindhoven, The Netherlands

Dynamic admission control strategies are of increasing importance as revenue management tools in service and manufacturing systems. In telecommunications and in particular in telephone service and support applications, such strategies are commonly used in order to increase flexibility in the allocation of resources among different customer types. In this paper we consider the problem of dynamic admission control in a two class loss Markovian queueing system with different service rates for the two customer classes. We prove that an optimal admission policy can be generally described as follows. An arriving customer is admitted to the system if and only if the number of available servers exceeds a certain threshold, which depends on the number of customers of both classes already being served. In addition we develop a set of sufficient conditions which ensure that a customer class is “preferred”, in the sense that its customers are always admitted if there are free servers, regardless of the system congestion level.
1 \textbf{INTRODUCTION}

Dynamic admission control is of increasing importance for revenue management in service and manufacturing systems. In telecommunications and in particular in telephone service and support applications, such strategies are commonly used in order to increase flexibility in the allocation of resources among job types.

In a queueing context, a static admission rule specifies a priori whether each job class is admitted for service, based on the revenue that jobs of this class generate and their expected service requirements. This specification is made independently of the system state and is equivalent to determining whether one job class is preferred over the other. On the other hand, a dynamic admission policy offers increased flexibility because it makes the decision of admitting an arriving job contingent on the current level of congestion upon arrival, in addition to the job’s class. A dynamic admission policy is equivalent to a resource idling rule. Indeed, when a job is not admitted for service, it is effectively decided to keep a server idle in anticipation of future admission of more profitable jobs that would otherwise be lost.

In this paper we consider the problem of dynamic admission control in a two class loss Markovian queueing system with different service rates for the two job classes. More explicitly, the system we study has $s$ identical parallel servers, no waiting room and two classes of jobs. Class-$i$ jobs arrive according to a Poisson process with rate $\lambda_i$ and demand an exponential service time with mean $1/\mu_i$. If a class-$i$ job is admitted, a reward of $r_i > 0$ is gained after its service is finished. Our objective is to find dynamic admission policies that maximize the total expected discounted revenue over a finite or infinite horizon as well as the long-run average revenue. We develop sets of sufficient conditions which ensure that a job class is “preferred”, in the sense that its jobs are always admitted if there are free servers, regardless of the congestion level. Moreover, the optimal admission policy is of threshold type. We show that the thresholds are monotone under very restrictive conditions. Finally, we discuss several issues on determining a preferred class.

There has been an increasing interest in multiclass loss networks due to the growth of telecommunications systems. Admission control is a main focus of research
on loss networks: Ross [13] defines a general stochastic knapsack which consists of $s$ resource units to give service to jobs of $K$ classes. Each class is distinguished by its size, its arrival rate and its mean service time. Arrival and service rates may depend on the state of the system. The system we describe above is a stochastic knapsack with 2 classes, each having a size of 1, so that $K = 2$ and arrival process and service times are independent of the state of the system. Chapter 4 of Ross [13] gives a comprehensive review on the admission control problems of generalized stochastic knapsacks. The studies included in Ross [13] analyze the optimal policies which belong to certain classes of policies, namely coordinate convex, complete partitioning and trunk reservation policies. We, on the other hand, focus on the structural properties of a global optimal policy. An example we present in section 4 shows that optimal admission policies for the system under consideration are not necessarily of the types considered by Ross [13].

There have been earlier studies which investigate the structural properties of optimal admission policies for certain stochastic knapsacks: Miller [8] has studied a special case of the model described above, with $\mu_1 = \mu_2$ and $r_1 > r_2$, where the optimal policy is a trunk reservation policy so that it is optimal to accept class-1 jobs whenever there is an idle server and to admit class-2 jobs if and only if the total number of jobs in the system is below a certain threshold. Thus, the optimal policy in Miller’s work is a threshold type policy with a preferred class. Recently, Lewis, Ayhan and Foley [6] generalized this result to the system with a finite waiting room. Lippmann and Ross [7] analyze the optimal admission rule for a system with one server and no waiting room which receives offers from jobs according to a joint service time and reward probability distribution (this model is usually referred to as the streetwalker’s dilemma).

Harrison [3] shows that the $r\mu$ rule is optimal for a scheduling problem in a single server queue with two classes of jobs characterized by Poisson arrivals with rates $\lambda_i$, exponential service times with different rates $\mu_i$, and lump rewards, $r_i$, collected at the end of service. Thus, in order to maximize the expected present value of service rewards incurred over an infinite horizon, jobs who bring higher
average reward, i.e., $r_i \mu_i$, have to be scheduled first on the single server. One might expect that the priority rule in the scheduling system may be used to determine a preferred class in the loss system, but this is not the case in general as we show in a counterexample in Section 5. It should be noted that the term “preferred” is used herein to denote a class whose jobs are always admitted to the system when there is at least one free server, so theoretically there could be 0, 1 or 2 preferred class(es). This terminology reflects the preferential treatment that a class may enjoy in the sense that it is never rejected unless the system is full. It is therefore a global state independent property. With a different service rate and reward for each class, it is not clear whether a preferred class exists and, if it does, how to determine it. One additional difficulty stems from the fact that availability of jobs is also of concern since we have a loss system. Thus, the arrival rates also affect the determination of preferred class(es) as shown by examples in section 5. We also note that in the studies of Miller [8], Lewis et al. [6] and Lippmann and Ross [7], there exists a preferred class which is determined by the “$r \mu$ rule”.

We show that under certain conditions, a preferred class exists, and under a stronger set of conditions, the $r \mu$ rule determines a preferred class(es) correctly.

Altman, Jiménez and Koole [1], Koole [4] and Örmeç [10] prove the existence of an optimal admission policy that is characterized by acceptance thresholds for both classes for the system we analyze in this paper. Specifically, for class-2 jobs, there exist thresholds such that the optimal policy accepts class-2 jobs only if the number of class-1 jobs in the system is less than a specified threshold, and rejects otherwise. Similarly, the thresholds for class 1 are expressed in the number of class-2 jobs already in the system. We present this result to be complete in our analysis, and then we show the monotonicity of thresholds under very restrictive conditions. The systems considered in Altman et al. [1] and Koole [4] are more general systems than the one we present here; however, they do not have any results regarding to the monotonicity of thresholds and preferred class(es).

There has been more research on loss systems lately: Ku and Jordan [5] consider two stations in tandem each with no waiting room and parallel servers.
Carrizosa, Conde and Munoz-Marquez [2] present an optimal static control policy for acceptance/rejection of $k$ classes in an $M/G/s/s$ queue. In all these studies, there exists a preferred class which is determined by a variation of the $r\mu$ rule.

This paper is organized as follows: In the next section, we develop a Markov Decision Process (MDP) model for the system described above. In Section 3 we present sets of sufficient conditions for each class to be preferred. Section 4 presents the existence of an optimal threshold policy, and shows that under certain conditions the optimal acceptance thresholds have intuitive monotonicity properties. Section 5 discusses the issues regarding preferred classes with several examples, and also includes a counterexample to the $r\mu$ rule. Section 6 discusses extensions.

2 MARKOV DECISION MODEL OF THE SYSTEM

2.1 Discrete Time Model of the System

In this section, we build a discrete time Markov decision process (MDP) for the system described above with the objective of maximizing total expected discounted returns over a finite time horizon with $\beta$ as the discount rate. We define the state of the system as $x = (x_1, x_2)$, where $x_i$ is the number of class-$i$ jobs. We interpret discounting as exponential failures, i.e., the system closes down in an exponentially distributed time with rate $\beta$ (for the equivalence of the processes with discounting and without discounting but with an exponential deadline, see e.g., Walrand [14]).

We also assume, without loss of generality, $\mu_1 < \mu_2$ (Note that the case $\mu_1 = \mu_2$ was covered by Miller [8]). Then, the maximum possible rate out of any state is $\lambda_1 + \lambda_2 + s\mu_2 + \beta$. Since the time between transitions is always exponentially distributed and the maximum rate of transitions is finite, we can use uniformization and normalization to build a discrete time equivalent of the original system. We assume, using the appropriate time scale, $\lambda_1 + \lambda_2 + s\mu_2 + \beta = 1$ so that the system will be observed at exponentially distributed intervals with mean 1. There will be an arrival with probability $\lambda_1 + \lambda_2$ and a potential service completion with probability $s\mu_2$ so that a real service completion due to a class-$i$ job occurs with probability
The assumption $\mu_1 < \mu_2$ implies that class-1 jobs are “slow” jobs. We use this assumption quite often to couple the service times of class-1 and class-2 jobs in such a way that if the coupled class-1 job finishes its service, then the coupled class-2 job also completes its service with probability 1. In terms of discrete time, this translates to the following: Both jobs leave the system with probability $\mu_1$, and a class-2 job departs from the system with probability $\mu_2 - \mu_1$ leaving the coupled class-1 job in the system. Thus, coupling never allows a coupled class-1 job to leave the system while the coupled class-2 job is still there.

We also define the state of the system in a little different way, i.e., by including the last event occurred: If there is a potential service completion, we denote the system state by $x = (x_1, x_2)$, where $x_i$ is the number of class-$i$ jobs. If there is a class-$j$ arrival, then the state is $(x; j) = (x_1, x_2; j)$ so that there are $x_i$ class-$i$ jobs in the system and a class-$j$ job has just arrived. Note that we always have $x_1 + x_2 \leq s$ and the actions are defined only for the states corresponding to an arrival.

### 2.2 Markov Decision Model for Finite Horizon

Define $u_n(x)$ ($v_n(x; j)$) to be the maximal expected $\beta$-discounted reward for the system starting in state $x$ ($(x; j)$) when $n$ observation points remain in the horizon. Let $S$ be the set on which $u_n$’s are defined, i.e., $S = \{x : x_1 + x_2 \leq s\}$. We define $a_n(x; j)$ as the optimal action in state $(x; j)$ when there are $n$ more transitions. $a_n(x; j)$ is set to 1 if it is optimal to accept the arriving job of class $j$ and to 0 otherwise. We note that $a_n(x; j)$ is defined as 1 if both accepting and rejecting are optimal. Now we present the optimality equations. With $e_j$ as the vector which has a 1 at the $j$th coordinate, and 0 elsewhere, we have:

\[
\begin{align*}
v_n(x; j) & = \max\{u_n(x + e_j), u_n(x)\}, \quad j = 1, 2, \text{ for } x_1 + x_2 < s, \text{ and } \quad (1) \\
v_n(x; 1) & = v_n(x; 2) = u_n(x), \text{ so that } a_n(x; 1) = a_n(x; 2) = 0, \text{ for } x_1 + x_2 = s,
\end{align*}
\]

where for $x_1 + x_2 \leq s$:

\[
u_{n+1}(x) = x_1\mu_1r_1 + x_2\mu_2r_2 + \lambda_1v_n(x; 1) + \lambda_2v_n(x; 2) + \]

\[
\quad + s\mu_2 - x_1\mu_1 - x_2\mu_2.
\]
\[ x_1 \mu_1 u_n(x - e_1) + x_2 \mu_2 u_n(x - e_2) + (s \mu_2 - x_1 \mu_1 - x_2 \mu_2) u_n(x). \]

Note that the last term in (2) refers to fictitious service completions. Further, it is notationally convenient to define \( u_n(-1, x_2) = u_n(0, x_2) \) and \( u_n(x_1, -1) = u_n(x_1, 0) \).

### 2.3 Infinite Horizon Models

We prove all our results for the objective of maximizing the total expected \( \beta \)-discounted reward for a finite number of transitions, \( n \), including the “fictitious” transitions due to the “fictitious” service completions (last term in (2)). Thus, “finite” horizon problems are pseudo finite problems. We use the powerful tool of induction to prove our results for all finite \( n \). All the results proven for finite \( n \) are true for the limit \( n \to \infty \), so the corresponding conclusions are valid when total expected \( \beta \)-discounted reward over an infinite horizon is maximized. Moreover, since the state space and the action space in each state are finite, the results also hold for \( \beta = 0 \), so we have the same conclusions for the long-run average reward. Here, we note that for the results regarding to preferred class, we specify the initial value function \( u_0 \) in such a way that the rewards of jobs, who are still in the system at \( n = 0 \), are collected even if their services have not been finished. Of course, this makes no difference in the optimal policy for infinite horizon problems.

We define \( u(x) (v(x; j)) \) as the maximal expected \( \beta \)-discounted reward for the system starting in state \( x \) ((\((x; j))\)) over an infinite horizon. Thus, for \( \beta > 0 \), we have:

\[
v(x; j) = \lim_{n \to \infty} v_n(x; j) \text{ and } u(x) = \lim_{n \to \infty} u_n(x)
\]

\( a(x; j) \) is the corresponding action in state \((x; j)\) so that \( a(x; j) = 1 \) if the job is accepted and \( a(x; j) = 0 \) otherwise. For \( \beta = 0 \), \( u(x) \to \infty \), so we need to consider the relative value functions and the gain in the usual MDP formulation.

### 2.4 Effect of an additional job

We define \( D_n(ij)(x) \) as the difference in the total expected \( n \)-period discounted rewards between system A and system B if system A starts in state \( x \) ‘plus’ one
class-$i$ job and system B starts in $x$ plus a class-$j$ job. We, occasionally, drop the arguments $x$ and $n$ later on, when there is no danger of confusion in the reference. The four $D_n(ij)$ functions of interest are $D_n(10)$, $D_n(20)$, $D_n(12)$ and $D_n(21)$. It is easy to see that $D_n(10)(x) = u_n(x+e_1) - u_n(x)$, $D_n(20)(x) = u_n(x+e_2) - u_n(x)$ and $D_n(12)(x) = -D_n(21)(x) = u_n(x+e_1) - u_n(x+e_2)$. We can interpret the difference $D_n(j0)(x)$ as the net benefit of the system due to an additional class-$j$ job in state $x$ when there are $n$ more transitions, whereas $D_n(12)(x)$ is the net benefit to the system when in state $x+e_2$ one class-2 job is changed to a class-1 job.

Recall that if both rejecting and accepting a job is optimal, then we choose to accept it. So we have:

$$a_n(x; j) = 1 \iff 0 \leq D_n(j0)(x)$$  \hspace{1cm} (3)

Thus, the optimal policy accepts a class-$j$ job only if the benefit it brings to the system is non-negative. Moreover, if the net benefit of accepting a class-$j$ job is non-negative for all states, then class $j$ is a preferred class, i.e., if $D_n(j0)(x) \geq 0$ for all $x \in S$ and for all $n$, then class-$j$ jobs are preferred.

2.5 A remark on rewards

In this model, we consider only the rewards collected at the end of service. Rejection costs, say $b_i$, which are incurred at the time of the arrival of a rejected job could be incorporated in the model by redefining the reward $r_i$ as $r_i + \frac{\mu_i + \beta}{\mu_i} b_i$. Örmeci [10] gives all the equivalent results of this paper in terms of both rejection costs, $b_i$, and rewards, $r_i$, which are notationally more complicated although the methods of proofs with or without rejection costs are the same.

The present value of the reward brought by a class-$i$ job is $\frac{r_i \mu_i}{\mu_i + \beta}$ due to the discounting. We refer to this quantity as the immediate reward of a class-$i$ job and denote it by $R_i$. Thus, $r_i$ is the value of the reward in the end of service, whereas $R_i$ is its value in the beginning of the service. For $\beta = 0$, we have $R_i = r_i$. Another quantity of interest is the average reward, or reward rate, of a class-$i$ job, $r_i \mu_i$. 
3  EXISTENCE OF A PREFERRED CLASS

In this section, we show that under certain conditions jobs of one class are admitted to the system whenever there is an idle server; i.e., there exists a preferred class. In determining preferred class(es), the first criterion one thinks of is the \( r \mu \) rule derived from the results of Harrison [3]: His results establish the \( r \mu \) rule as the optimal policy for a scheduling problem in a single server queue with two classes of jobs characterized by Poisson arrivals with rates \( \lambda_i \), exponential service times with different rates \( \mu_i \), and lump rewards, \( r_i \), collected at the end of service. Therefore, in this scheduling system jobs who bring higher average reward, since \( r_i \mu_i \) is the average reward per class-\( i \) job, have priority over the others. Now we note that the characteristics of jobs in Harrison [3] are identical to those in our model. Thus, we can expect that the optimal priority rule in the scheduling system may be used to determine a preferred class in the loss system: This corresponds to admitting all jobs of the class which brings the highest average reward, \( r_i \mu_i \), whenever there is at least one idle server. We will see that under certain conditions, a preferred class is, indeed, determined by the \( r \mu \) rule. However, this rule does not hold for all parameters; we present a counterexample in section 5. Another criterion to specify a preferred class can be the index rule which uses the immediate reward of each class, \( R_i \); but it is easy to see that if class-1 jobs are slow enough, the optimal policy may reject them even if their immediate rewards are high. As we see later in this section, each of the criteria favors one of the classes, and neither of them determines a preferred class for all possible values.

It will be convenient to present the conditions for class 1 to be preferred in terms of average rewards, \( r_i \mu_i \)'s, and for class 2 in terms of immediate rewards, \( R_i \)'s. The strength of class-1 jobs is the steady returns they bring to the system because of the longer service times: Consider a firm which has two kinds of jobs, one of which brings $12,000 of profit for a project of 12 months whereas the other requires a service of 3 months for $3,000. Thus, the average profits are $1,000 and $1,200 respectively. If the arrival rates for both classes are very low, then the firm will always accept class-1 jobs since they guarantee that a server will generate reward
for a longer period. If the arrival rates are very high, the system is willing to wait for class-2 jobs.

The main results of this section are summarized in two theorems, each presenting a condition for one of the classes to be preferred. From (3), we know that if $D_n(i0)(x) \geq 0$ for all $x \in S$ and for all $n \geq 0$, then class $i$ is preferred for all $n \geq 0$. Therefore, we are interested in finding non-negative lower bounds on $D_n(i0)(x)$ for all $x \in S$ and for all $n \geq 0$. In the proofs, this requires to consider not only the effect of an additional class-$i$ job, i.e., $D_n(i0)(x)$, but also the effect of changing a class-$i$ job to a class-$j$ job, i.e., $D_n(ij)(x)$. Thus, we will present upper and lower bounds on $D_n(10)(x)$, $D_n(20)(x)$ and $D_n(12)(x)$. Note that an upper (lower) bound on $D_n(12)(x)$ is a lower (upper) bound on $D_n(21)(x)$.

We derive explicit expressions for these bounds by using sample path arguments on the value functions corresponding to discounted finite horizon problems. The proofs for the validity of these bounds will require induction, so we specify the initial value function, $u_0$, which we have mentioned briefly in Section 2, as follows:

$$u_0(x) = x_1R_1 + x_2R_2 \quad \forall x \in S. \quad (4)$$

This function corresponds to the assumption that the later rewards of jobs, who are still in the system at $n = 0$, are collected at $n = 0$.

In section 3.1, we give upper bounds on $D_n(i0)(x)$ and $D_n(12)(x)$, which are valid for all possible parameters. Section 3.2 presents non-negative lower bounds for $D_n(20)(x)$ based on the upper bound on $D_n(12)(x)$ from section 3.1, which immediately gives a condition for class 2 to be preferred. In section 3.3, we derive lower bounds on $D_n(10)(x)$ and $D_n(12)(x)$, from which we conclude that under some conditions class 1 is preferred.

### 3.1 Upper bounds on the differences $D_n(i0)(x)$ and $D_n(12)(x)$

These bounds are relatively easier to obtain mainly because they are valid for all ranges of parameters. Moreover, the sample path argument which gives the expressions for the bounds can be used directly to prove their validity:
Lemma 1 For all $x \in S$ and for all $n \geq 0$:

(i) $D_n(i0)(x) \leq R_i$ for $i = 1, 2$.

(ii) $D_n(12)(x) = -D_n(21)(x) \leq R_1 - R_2$.

Proof. We prove the statements by a sample path analysis.

(1) Assume that system A is in state $x + e_i$ and system B in $x$ in period $n$. We let system A follow the optimal policy, and system B imitate all the decisions of system A. We couple the two systems via the service and interarrival times, i.e., except for the additional job in system A, all the departure and arrival times are the same in both systems. Note that system B can always imitate system A since it always has at least as many free servers as system A does. Then, the difference in the expected returns of systems A and B is only due to the additional job in A:

$$D_n(i0)(x) = u_n(x + e_i) - u_n(x) \leq u_n(x + e_i) - u_n^B(x) = R_i.$$ 

where $u_n^B(x)$ is the expected discounted return of system B and $R_i$ is the immediate reward of the additional class-$i$ job in the system, which will be collected eventually due to the definition of $u_0$.

(2) Assume that system A starts in state $x + e_1$ and system B in $x + e_2$. We couple the additional class-1 job, say job $j_1$, in system A with the additional class-2 job, say job $j_2$, as well as all other service and interarrival times, so that, as discussed earlier, if $j_1$ leaves the system, $j_2$ also leaves. Let system A follow the optimal policy and B imitate all decisions of A. Then the difference in the expected discounted returns of A and B is only due to the additional jobs at the start:

$$D_n(12)(x) = u_n(x + e_1) - u_n(x + e_2) \leq u_n(x + e_1) - u_n^B(x + e_2) = R_1 - R_2,$$

with $u_n^B(x + e_2)$ the expected discounted return of system B. \qed

3.2 A sufficient condition for class 2 to be preferred

The first part of this section is devoted to the derivation of non-negative lower bounds on $D_n(20)(x)$. Then we will give a detailed proof for the validity of these
bounds. We conclude with a theorem, which establishes a sufficient condition for class-2 customers to be preferred.

Let $d_2$ be a lower bound on $D_n(20)(x)$ for all $x \in \mathcal{S}$ and for all $n \geq 0$, so that $d_2 \leq \min_{x \in \mathcal{S}} \{D_n(20)(x)\}$. We will first find a sufficient condition for $d_2$ to be a non-negative lower bound, then we will define $\bar{d}_2$, a kind of maximal $d_2$, that can be obtained from this condition.

Now consider the discounted infinite horizon problem. We couple two systems: Let system A be in state $x + e_2$ and system B in $x$. System A follows the not necessarily optimal strategy and rejects all jobs in the next transition if the transition is an arrival, and follows the optimal policy afterwards, whereas system B always takes the optimal actions. Consider an arrival. If system B also rejects either of the two classes, both systems remain in their current states, preserving the extra class-2 job which leads to a minimum difference of $d_2$ in the rewards of the two systems by definition of $d_2$. Acceptance of a class-1 job to system B leads two systems to two different states $x + e_2$ and $x + e_1$. If a class-2 job is admitted to system B, then the two systems couple with no difference in reward. With the departure of the additional class-2 job in system A, the systems again enter the same state but with a return of $r_2$, whereas all other service completions keep the extra class-2 job in system A so that the difference between two systems is at least $d_2$ by its definition.

If we let $u_{n+1}^A$ be total expected discounted reward of system A, then:

$$D_{n+1}(20)(x) = u_{n+1}(x + e_2) - u_{n+1}(x) \geq u_{n+1}^A(x + e_2) - u_{n+1}(x) \geq \lambda_1 \min \{D_n(21)(x), D_n(20)(x)\} + \lambda_2 \min \{0, D_n(20)(x)\} + \mu_2 r_2$$

$$+ (s - 1)\mu_2 \min_{y \in \mathcal{S}} \{D_n(20)(y)\}$$

$$\geq \lambda_1 \min \{R_2 - R_1, d_2\} + \lambda_2 \min \{0, d_2\} + \mu_2 r_2 + (s - 1)\mu_2 d_2.$$ 

where the first inequality is by definition of the policy that system A follows, the second one is due to the coupling described above, and the last one follows from part (ii) of Lemma 1 and definition of $d_2$. From this, we can derive the following inequality, which assures that $d_2$ is a lower bound on $D_n(20)(x)$ for all $x \in \mathcal{S}$:

$$\lambda_1 \min \{R_2 - R_1, d_2\} + \lambda_2 \min \{0, d_2\} + \mu_2 r_2 + (s - 1)\mu_2 d_2 \geq d_2.$$

(5)
The case \( d_2 < 0 \) is not interesting to us for the time being, since it will not give any information about class 2 to be preferred. Thus, we now assume \( d_2 \geq 0 \). Then the above inequality reduces to the following:

\[
\lambda_1 \min\{R_2 - R_1, d_2\} + \mu_2 r_2 \geq (\lambda_1 + \lambda_2 + \mu_2 + \beta)d_2,
\]

by uniformization. Now we have two cases:

**Case 1**  \( d_2 \leq R_2 - R_1 \): Then inequality (6) becomes:

\[
\frac{\mu_2 r_2}{\lambda_2 + \mu_2 + \beta} \geq d_2.
\]

Define

\[
\tilde{d}_2 = \frac{\mu_2 r_2}{\lambda_2 + \mu_2 + \beta},
\]

then \( \tilde{d}_2 \) is always positive. Recall that this expression is a valid lower bound for \( D_n(20)(x) \) only if \( \tilde{d}_2 \leq R_2 - R_1 \). It is easy to see that:

\[
\frac{\lambda_2 + \mu_2 + \beta}{\lambda_2} \leq \frac{R_2}{R_1} \iff \frac{\mu_2 r_2}{\lambda_2 + \mu_2 + \beta} = \frac{R_2(\mu_2 + \beta)}{\lambda_2 + \mu_2 + \beta} \leq R_2 - R_1. \tag{7}
\]

**Case 2**  \( d_2 \geq R_2 - R_1 \): Then, we have from inequality (6):

\[
\lambda_1(R_2 - R_1) + (\mu_2 + \beta)R_2 \geq (\lambda_1 + \lambda_2 + \mu_2 + \beta)d_2.
\]

Define:

\[
\tilde{d}_2 = \frac{(\lambda_1 + \mu_2 + \beta)R_2 - \lambda_1 R_1}{\lambda_1 + \lambda_2 + \mu_2 + \beta}.
\]

Now, it is easy to check the following:

\[
\frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} \leq \frac{R_2}{R_1} \iff \tilde{d}_2 \geq 0 \quad \& \quad \frac{R_2}{R_1} \leq \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2} \iff \tilde{d}_2 \geq R_2 - R_1. \tag{8}
\]

The following lemma summarizes the results of our derivation above:

**Lemma 2**  For all \( n \geq 0 \) and for all \( x \in S \):

(i) if \( \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2} \leq \frac{R_2}{R_1} \), \( \frac{r_2 \mu_2}{\lambda_2 + \mu_2 + \beta} \leq D_n(20)(x) \), and

(ii) if \( \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} \leq \frac{R_2}{R_1} \leq \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2} \), \( \frac{(\lambda_1 + \mu_2 + \beta)R_2 - \lambda_1 R_1}{\lambda_1 + \lambda_2 + \mu_2 + \beta} \leq D_n(20)(x) \).
Proof. The proof of the lemma merely repeats the previous derivation. For the induction proofs of the two cases, we only have to verify the inequalities for \( n = 0 \).

According to the definition of \( u_0 \) in (4), we have \( D_0(20)(x) = R_2 = \frac{r_2\mu_2}{\mu_2 + \beta} \), so

\[
\begin{align*}
(i) & \quad \frac{r_2\mu_2}{\lambda_2 + \mu_2 + \beta} \leq \frac{r_2\mu_2}{\mu_2 + \beta} = R_2 = D_0(20)(x). \\
(ii) & \quad \frac{(\lambda_1 + \mu_2 + \beta)R_2 - \lambda_1 R_1}{\lambda_1 + \lambda_2 + \mu_2 + \beta} < \frac{r_2\mu_2}{\mu_2 + \beta} = R_2 = D_0(20)(x). \quad \Box
\end{align*}
\]

Since both lower bounds on \( D_n(20)(x) \) are non-negative in Lemma 2 we immediately have the following result:

**Theorem 1** If \( R_2 \geq \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} R_1 \), then class 2 is a preferred class for all \( n \geq 0 \).

We summarize the intuition behind this result: Accepting a class-2 job brings a delayed reward of \( R_2 \). If there is a class-1 arrival before the departure of the class-2 job and before the exponential failure of the system, which happens with probability \( \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} \), then the system loses a reward of \( R_1 \). Thus, if the guaranteed reward due to a class-2 job is higher than the possible future reward of a class-1 job, it is optimal to accept the class-2 job. Theorem 1 also implies:

**Corollary 1** If \( r_2 \geq r_1 \), so that class-2 jobs bring higher rewards and require shorter service times, class-2 jobs are preferred.

**Remark 1** This corollary also holds for general service time distributions if the service time for class-2 jobs is stochastically smaller than for class 1. The proof follows from a straightforward sample path argument.

### 3.3 A sufficient condition for class 1 to be preferred

We will derive lower bounds on \( D_n(10)(x) \) and \( D_n(12)(x) \), which is somewhat more complex than the previous section since we have to consider both bounds at the same time. However, we will only outline the proof for their validity since the method of the proof is the same as in Lemma 2. Finally, we will summarize the implications of these bounds in terms of class 1 to be preferred with a theorem.
Let $d_1$ and $d_{12}$ be lower bounds for $D_n(10)(x)$ and $D_n(12)(x)$ for all $x \in S$ and for all $n$, respectively. We will consider two pairs of coupled systems, one for $D_n(10)(x)$ and the other for $D_n(12)(x)$ to derive two inequalities from which we can compute maximal $\bar{d}_1$, and $\bar{d}_{12}$. These expressions will be used also in the next section to prove concavity of the value functions under certain conditions.

Now consider the first pairs. Assume that system A is in state $x + e_1$ and system B is in $x$, and couple the two systems in such a way that A rejects all jobs in the next transition and then continues with the optimal policy, whereas system B follows the optimal policy in all transitions. If upon an arrival system B also rejects either of the two classes, both systems remain in their current states, preserving the extra class-1 job which leads to a minimum future difference of $d_1$ in the rewards of the systems due to the definition of $d_1$. Acceptance of a class-1 job to system B leads both systems to enter the same state with no difference in reward. If a class-2 job is admitted to system B, then the systems move to two different states $x + e_1$ and $x + e_2$. With the departure of the additional class-1 job in system A, the systems again enter the same state but with a return of $r_1$, whereas for all other service completions difference between the two systems remains the same, i.e., the extra class-1 job, so at least $d_1$. Letting $u^A_{n+1}$ be total expected discounted reward of system A, we have:

$$D_{n+1}(10)(x) = u_{n+1}(x + e_1) - u_{n+1}(x) \geq u^A_{n+1}(x + e_1) - u_{n+1}(x)$$

$$\geq \lambda_1 \min \{0, D_n(10)(x)\} + \lambda_2 \min \{D_n(12)(x), D_n(10)(x)\} + \mu_1 r_1$$

$$+ (s \mu_2 - \mu_1) \min_{y \in S} \{D_n(10)(y)\}$$

$$\geq \lambda_1 \min \{0, d_1\} + \lambda_2 \min \{d_{12}, d_1\} + \mu_1 r_1 + (s \mu_2 - \mu_1)d_1.$$

The second inequality is due to coupling, and the third follows from $d_1$ and $d_{12}$ being lower bounds. We are interested in a non-negative $d_1$, so for the time being we assume $d_1 \geq 0$. Then for $d_1$ to be a lower bound for $D_{n+1}(10)(x)$, we require:

$$\lambda_2 \min \{d_{12}, d_1\} + \mu_1 r_1 + (s \mu_2 - \mu_1)d_1 \geq d_1.$$  \hspace{1cm} (10)

Now consider the second pair of systems. Let system $A'$ be in state $x + e_1$ and system $B'$ in $x + e_2$. System $B'$ takes the optimal actions and system $A'$ imitates all
the actions of system B' in the next transition and afterwards follows the optimal policy. As before, we couple the additional class-2 job, say job $j_2$, in system B' with the additional class-1 job, say job $j_1$ in system A', as well as all other service and interarrival times. So, if $j_1$ leaves the system, which happens with probability $\mu_1$, $j_2$ also leaves. The departure of $j_1$ leads the system to couple with a reward of $r_1 - r_2$, the departure of $j_2$ alone, which happens with probability $\mu_2 - \mu_1$, takes the systems to two different states, $x + e_1$ and $x$ with a reward of $-r_2$. If there is any other transition, both systems continue to have their additional jobs. So for the difference between the two systems due to changing a class-1 job to class 2, we have:

$$D_{n+1}(12)(x) \geq \mu_1(r_1 - r_2) + (\mu_2 - \mu_1)(-r_2 + D_n(10)(x)) + (\lambda_1 + \lambda_2 + (s - 1)\mu_2) \min_{y \in S} \{D_n(12)(y)\}$$

$$\geq r_1\mu_1 - r_2\mu_2 + (\mu_2 - \mu_1)d_1 + (1 - \mu_2 - \beta)d_{12}$$

The first inequality is due to the coupling and the second follows from $d_1$ and $d_{12}$ being lower bounds. This leads us to the following inequality:

$$r_1\mu_1 - r_2\mu_2 + (\mu_2 - \mu_1)d_1 \geq (\mu_2 + \beta)d_{12} \quad (11)$$

In order to find the maximal $d_1$ and $d_{12}$, satisfying (10) and (11). We need to distinguish two cases due to inequality (10):

**Case 1.** $d_1 < d_{12}$: Then inequality (10) becomes:

$$r_1\mu_1 \geq (1 - \lambda_2 - s\mu_2 + \mu_1)d_1 = (\lambda_1 + \mu_1 + \beta)d_1,$$

where the equality follows from the uniformization. So define:

$$\bar{d}_1 = \frac{r_1\mu_1}{\lambda_1 + \mu_1 + \beta},$$

which is positive. The maximal $d_{12}$ is obtained from (11) as:

$$\bar{d}_{12} = \frac{r_1\mu_1(\lambda_1 + \mu_2 + \beta)}{(\mu_2 + \beta)(\lambda_1 + \mu_1 + \beta)} - \frac{r_2\mu_2}{\mu_2 + \beta}.$$

From these values it is easy to check:

$$\frac{r_2\mu_2}{r_1\mu_1} < \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta} \iff \bar{d}_1 < \bar{d}_{12}. \quad (12)$$
Case 2. \( d_1 \geq d_{12} \): Now inequality (10) becomes:

\[
\lambda_2 d_{12} + r_1 \mu_1 + (s \mu_2 - \mu_1) d_1 \geq d_1.
\] (13)

It can be shown that the maximal \( d_1 \) and \( d_{12} \) satisfy inequalities (11) and (13) with equality. Solving the following system of equations for \( \tilde{d}_1 \) and \( \tilde{d}_{12} \):

\[
\begin{align*}
r_1 \mu_1 &= (\lambda_1 + \lambda_2 + \mu_1 + \beta) \tilde{d}_1 - \lambda_2 \tilde{d}_{12} \\
r_1 \mu_1 - r_2 \mu_2 &= -(\mu_2 - \mu_1) \tilde{d}_1 + (\mu_2 + \beta) \tilde{d}_{12},
\end{align*}
\]

where we used \( \lambda_1 + \lambda_2 + c \mu_2 + \beta = 1 \) for the first equality, we obtain:

\[
\begin{align*}
\tilde{d}_1 &= \frac{r_1 \mu_1 (\lambda_2 + \mu_2 + \beta) - \lambda_2 r_2 \mu_2}{(\lambda_1 + \lambda_2 + \mu_2 + \beta)(\mu_1 + \beta) + \lambda_1 (\mu_2 - \mu_1)}, \quad \text{and} \\
\tilde{d}_{12} &= \frac{r_1 \mu_1 (\lambda_1 + \lambda_2 + \mu_2 + \beta) - r_2 \mu_2 (\lambda_2 + \mu_1 + \beta)}{(\lambda_1 + \lambda_2 + \mu_2 + \beta)(\mu_1 + \beta) + \lambda_1 (\mu_2 - \mu_1)}
\end{align*}
\]

From the values of \( \tilde{d}_1 \) and \( \tilde{d}_{12} \), it is easy to check that

\[
\begin{align*}
\frac{r_2 \mu_2}{r_1 \mu_1} \geq \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta} \iff \tilde{d}_1 \geq \tilde{d}_{12} \& \quad r_1 \mu_1 \geq \frac{\lambda_2}{\lambda_2 + \mu_2 + \beta} \tilde{d}_{12} \iff \tilde{d}_1 \geq 0,
\end{align*}
\]

where the first condition is complementary to (12), as expected.

The following lemma summarizes the lower bounds found in this section.

**Lemma 3** For all \( n \geq 0 \) and for all \( x \in S \):

- (i) if \( \frac{r_2 \mu_2}{r_1 \mu_1} \leq \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta} \), \( \frac{r_1 \mu_1}{\lambda_1 + \mu_1 + \beta} \leq D_n(10)(x) \), and
  \( \frac{r_1 \mu_1 (\lambda_1 + \mu_2 + \beta)}{(\mu_2 + \beta)(\lambda_1 + \mu_1 + \beta)} - \frac{r_2 \mu_2}{\mu_2 + \beta} \leq D_n(12)(x) \)

- (ii) if \( \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta} \leq \frac{r_2 \mu_2}{r_1 \mu_1} \leq \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2} \),
  \( \frac{r_1 \mu_1 (\lambda_2 + \mu_2 + \beta) - \lambda_2 r_2 \mu_2}{(\lambda_1 + \lambda_2 + \mu_2 + \beta)(\mu_1 + \beta) + \lambda_1 (\mu_2 - \mu_1)} \leq D_n(10)(x) \), and
  \( \frac{r_1 \mu_1 (\lambda_1 + \lambda_2 + \mu_2 + \beta) - r_2 \mu_2 (\lambda_1 + \lambda_2 + \mu_1 + \beta)}{(\lambda_1 + \lambda_2 + \mu_2 + \beta)(\mu_1 + \beta) + \lambda_1 (\mu_2 - \mu_1)} \leq D_n(12)(x) \)
Proof. For the induction proof, we only have to verify that \( \bar{d}_1 \) and \( \bar{d}_{12} \) are lower bounds for \( D_0(10)(x) \) and \( D_0(12)(x) \), respectively. With some elementary algebra, the four inequalities can be shown to hold for \( u_0 \) given by equation (4).

Under the given conditions both lower bounds on \( D_n(10)(x) \) are non-negative. Thus, this lemma immediately leads to:

**Theorem 2** If \( r_1 \mu_1 \geq \frac{\lambda_2}{\lambda_2 + \mu_2 + \beta} r_2 \mu_2 \), then class 1 is a preferred class for all \( n \geq 0 \).

**Remark 2** One may verify that for the one-server system Theorem 2 gives just the necessary and sufficient condition for the policy which accepts only class 2 not to be optimal.

**Remark 3** In all results of this section, the inequalities and the conditions on the parameters never include the number of servers. At first sight, this is surprising. However, it is not really so remarkable as all these results provide sufficient conditions for always accepting jobs of a class so that the main question is on the decisions when almost all servers, or more boldly all but one server, are busy.

## 4 ON AN OPTIMAL THRESHOLD POLICY

Intuitively, we expect that it should be less profitable to accept jobs when there are many jobs already in the system, and so the benefit to the system of additional jobs should decrease in the number of jobs in the system. In this section, we first present two results of Altman et al. [1], Koole [4] and Örmen [10] on the existence of an optimal threshold policy: Lemma 4 shows that the benefit due to an additional class-\( j \) job, i.e., \( D_n(j0)(x) \), is decreasing in the number of class-\( i \) jobs, \( x_i, i \neq j \), which guarantees the existence of an optimal threshold policy as stated in Theorem 3. We also expect that the benefit of an additional class-\( j \) job decreases in the number of class-\( j \) jobs; which corresponds to the concavity of \( u_n \) in \( x_j \) for fixed \( x_i, i \neq j \). In the end of this section, we prove the concavity of \( u_n \) in \( x_2 \) for fixed \( x_1 \) under very restrictive conditions.
Lemma 4 For all $x$ with $x_1 + x_2 + 2 \leq s$ (or equivalently for all $x + e_1 + e_2 \in S$):

$$u_n(x + e_1 + e_2) - u_n(x + e_2) - u_n(x + e_1) + u_n(x) \leq 0 \quad \forall n \geq 1,$$

whenever the inequality is true for $n = 0$.

To see the intuition in (14) we write it as:

(i) $u_n(x + e_1 + e_2) - u_n(x + e_1) \leq u_n(x + e_2) - u_n(x)$ &

(ii) $u_n(x + e_1 + e_2) - u_n(x + e_2) \leq u_n(x + e_1) - u_n(x)$.

Thus, (i) says that an additional class-2 job brings more benefits in state $x$ than it does in state $x + e_1$, and (ii) says that an additional class-1 job is more interesting in state $x$ than in state $x + e_2$. This guarantees an optimal threshold policy:

Theorem 3 There exist numbers $\{l_1^n(0), ..., l_1^n(s - 1)\}_{i=1}^2$ such that $a_n(x; 1) = 0$ if $x_2 \geq l_1^n(x_1)$ and 1 otherwise, and similarly, $a_n(x; 2) = 0$ if $x_1 \geq l_2^n(x_2)$, and 1 otherwise.

In addition to equation (14), what we expect and what we have observed in all examples is the following:

$$u_n(x + 2e_i) - u_n(x + e_i) \leq u_n(x + e_i) - u_n(x) \quad \forall n \geq 1,$$

which is the concavity of $u_n$ in $x_i$, and this would also imply the monotonicity of the thresholds. However, it is difficult to prove the concavity in full generality because of the boundary effects and state-dependent service rates. Using the bounds on the differences due to an additional job, $D_n(j0)(x)$, and due to changing the class of a job, $D_n(ij)(x)$, we can show the concavity of $u_n$’s in $x_2$ only under some restrictive conditions on the parameters:

Concavity condition 1: (a) $\frac{r_2\mu_2}{r_1\mu_1} \leq \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta}$ &

(b) $\lambda_1(\mu_2 - \mu_1) \leq (\mu_1 + \beta)(\mu_2 + \beta) \left(1 + \frac{\mu_2 + \beta}{\lambda_1}\right)$

Concavity condition 2: (a) $\frac{r_2\mu_2}{r_1\mu_1} \leq \frac{\lambda_1}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)}$ &

(b) $\lambda_1(\mu_2 - \mu_1) \leq (\mu_1 + \beta)(\mu_2 + \beta) \left(1 + \frac{\mu_2 + \beta}{\lambda_1 + \lambda_2}\right)$
First observe that under both conditions class 1 is a preferred class. For concavity condition 1 (a), this follows from Theorem 2, since
\[
\frac{r_2\mu_2}{r_1\mu_1} \leq \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta} \leq 1 \leq \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2}.
\]

Now consider concavity condition 2 (b). It can be shown that this inequality is equivalent to:
\[
\frac{\lambda_1}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)} + \frac{(\mu_2 + \beta)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \mu_2 + \beta)} \leq 1 + \frac{\mu_2 + \beta}{\lambda_1 + \lambda_2} \leq 1 + \frac{\mu_2 + \beta}{\lambda_2}.
\]
But then we have:
\[
\frac{\lambda_1}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)} + \frac{(\mu_2 + \beta)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \mu_2 + \beta)} \leq 1 + \frac{\mu_2 + \beta}{\lambda_1 + \lambda_2} \leq 1 + \frac{\mu_2 + \beta}{\lambda_2}.
\]
Thus when concavity condition 2 is satisfied, we have:
\[
r_2\mu_2 \leq \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2} r_1\mu_1
\]
which implies that class 1 is preferred by Theorem 2. Now we can present the following result:

**Lemma 5** Assume that either concavity condition 1 or concavity condition 2 is satisfied. Then, for all \( x \in S \) and for all \( n \geq 0 \), \( u_n(x) \) is concave in \( x_2 \) for each fixed \( x_1 \), i.e., \( D_n(20)(x + e_2) \leq D_n(20)(x) \).

The proof of this lemma is very technical, which requires to derive another explicit expression of a lower bound on \( D_n(20)(x) \) and use the exact values of lower bounds on \( D_n(12)(x) \) given in Lemma 3. Thus, we present the derivation of the new lower bound on \( D_n(20)(x) \) as well as the proof of this lemma in the Appendix. All these are shown by induction and the same proof techniques used in the previous section. Hence, all the material in the Appendix is conceptually straightforward although very tedious.

**Proposition 1** Assume that concavity condition 1 or concavity condition 2 holds. Then, \( l^n_2(k) \) is decreasing in \( k \).
Proof. By definition of $l_2^n(k)$, we have $a_n \left( l_2^n(k), k; 2 \right) = 0$. Then:

$$u_n \left( l_2^n(k), k + 2 \right) - u_n \left( l_2^n(k), k + 1 \right) \leq u_n \left( l_2^n(k), k + 1 \right) - u_n \left( l_2^n(k), k \right) < 0$$

where the first inequality follows from Lemma 5 and the second one is due to the fact that $a_n \left( l_2^n(k), k; 2 \right) = 0$ by definition of $l_2^n(k)$. So it is optimal to reject a class-2 job in state $(l_2^n(k), k + 1)$. Hence, by definition of $l_2^n(k + 1)$, we have $l_2^n(k + 1) \leq l_2^n(k)$. \hfill $\square$

Now we present an example to give an intuition about the thresholds, emphasizing the difference between our results and those in Ross [13]:

**Example 1** We consider the average rewards (so $\beta = 0$). We take $s = 11$. The other parameters are: $\lambda_1 = 15$, $\lambda_2 = 30$, $\mu_1 = 1$ $\mu_2 = 3$, $r_1 = 2$ and $r_2 = 1$. The optimal policy is given in Figure 1. In this example class 2 is preferred so that the thresholds for class 2 are rather trivial: $l_2(x_2) = s - x_2$ for all $x_2$. The thresholds for class 1 can be easily derived from Figure 1: $l_1(0) = 8$, $l_1(1) = 7$, $l_1(2) = 6$, $l_1(3) = 5$, $l_1(4) = 5$, $l_1(5) = 4$, $l_1(6) = 3$, $l_1(7) = 2$, $l_1(8) = 1$ and $l_1(9) = l_1(10) = 0$. 

![Figure 1: The optimal policy for the system in Example 1](image)
We have several observations: First, the thresholds are decreasing. Second, we would like to point out the difference of the optimal policy from three classes of policies considered in Ross [13], namely trunk reservation, complete partitioning and coordinate convex policies:

A trunk reservation policy is a policy determined by trunk reservation parameters, \( t_1 \) and \( t_2 \) so that \( a_n(x; i) \) is equal to 0 if \( x_1 + x_2 + 1 > s - t_i \), and \( a_n(x; i) = 1 \) otherwise. In this example, a class-1 job is accepted in state \((4,4)\) and rejected in state \((6,2)\), showing that the optimal policy is not a trunk reservation policy.

A complete partitioning policy is determined by positive integers, \( s_1 \) and \( s_2 \) with \( s_1 + s_2 \leq s \) such that \( a_n(x; i) = 0 \) if \( x_i + 1 \leq s_i \), and \( a_n(x; i) = 1 \) otherwise. This decomposes the whole stochastic knapsack to two smaller knapsacks with the \( i \)th knapsack dedicated to class-\( i \) with a capacity of \( s_i \). The optimal policy for this example is clearly not of complete partitioning type, since class-2 jobs can use all servers, and class-1 jobs can use up to 9 servers, depending on the current state of the system.

A coordinate convex policy is characterized by a coordinate convex set, which is any non-empty set \( \Omega \subseteq S \) such that if \( x \in \Omega \) and \( x_i > 0 \), then \( x - e_i \in \Omega \). Then \( a_n(x; i) = 0 \) if \( x + e_i \notin \Omega \), and \( a_n(x; i) = 1 \) otherwise. The optimal policy for this example is not a convex coordinate policy since state \((1,7)\) is accessible from state \((1,8)\) but not from \((0,8)\). In general, for systems with only class 2 as a preferred class, states \((x_1 + 1, t^l(x_1))\) can be reached from \((x_1 + 1, t^l(x_1) - 1)\), but not from \((x_1, t^l(x_1))\), when \( t^l(x_1) > 0 \), so that they are not coordinate convex. A similar argument is valid for systems with only class 1 preferred. Thus, whenever the optimal policy prefers exactly one class, it cannot be coordinate convex.
5 A DISCUSSION ON PREFERRED CLASS AND A COUNTEREXAMPLE

5.1 Systems with Two Preferred Classes

A policy which accepts both classes whenever there is an available server is called a complete sharing policy (see Ross [13]). The following result, which can be easily deduced from Theorem 1 and Theorem 2, gives a sufficient condition on the parameters of systems for which a complete sharing policy is optimal:

**Corollary 2** If \( \frac{\mu_2 + \beta}{\mu_1 + \beta} \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta} \leq \frac{r_2 \mu_2}{r_1 \mu_1} \leq \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2}, \) then it is optimal to accept all jobs whenever there is an idle server so that the optimal policy is complete sharing.

**Example 2** Consider a system with \( \beta = 0, \mu_1 = \mu_2 = 1, \) and \( \lambda_1 = \lambda_2 = 1. \) Then for all \( r_i \) with \( 0.5 < \frac{r_2}{r_1} < 2, \) it is optimal to accept both classes. So, independent of the number of servers, \( s, \) the range for the ratio of rewards is quite broad. Note that if \( s \) goes to infinity, it will always be optimal to accept both classes.

5.2 Effect of Arrival Rates on Preferred Class and a Counterexample to the \( r\mu \) Rule

The following example shows that the arrival rates play a major role in determining preferred class(es). Furthermore it provides a counterexample to the \( r\mu \) rule.

**Example 3** Consider a system with 6 servers for which \( \beta = 0, \mu_1 = 0.5, \mu_2 = 4, r_1 = 1.8 \) and \( r_2 = 0.255, \) so that \( r_1 \mu_1 = 0.9 \) and \( r_2 \mu_2 = 1.02. \) Optimal policies for three systems with different sets of arrival rates are shown in Figure 2.

For case (a) with \( \lambda_1 = 3 \) and \( \lambda_2 = 0.01, \) only class 1 is preferred even though \( r_1 \mu_1 < r_2 \mu_2, \) which is a counterexample to the \( r\mu \) rule. In case (b) with moderate arrival rates, i.e., \( \lambda_1 = \lambda_2 = 10, \) both classes are admitted to the system in all possible states. Finally (case (c)), if the class with the shorter jobs but higher average rewards arrive at the system often enough, it is optimal to wait for them in a number of states.
5.3 Question of Existence of a Preferred Class

In section 3, we gave sufficient conditions for each class to be preferred. These conditions are not complementary. Combining Theorems 1 and 2, we have:

**Corollary 3** Our results are inconclusive, if the parameter values satisfy:

\[
\frac{\lambda_2 + \mu_2 + \beta}{\lambda_2} < \frac{r_2 \mu_2}{r_1 \mu_1} < \frac{\lambda_1 (\mu_2 + \beta)}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)}. \tag{15}
\]

Take \(\beta = 0\). Then, unless \(\mu_1 << \lambda_1\) and \(\mu_1 << \mu_2\), the first term in (15) will be larger than the third implying that a preferred class does exist.

Whenever our results do not imply the existence of a preferred class, we know little about the optimal policy: We still have a threshold policy, and it is also easy to show that the optimal policy has to use all the servers. However, our results, so far, do not even guarantee that in each state at least one of the two classes has to be accepted. Rejecting both classes would be very counterintuitive. So we have the following conjecture:

**Conjecture 1** For all parameter values, there exists no state in which both classes are rejected.

If Conjecture 1 holds, so that in all states at least one of the classes is admitted to the system, then there can be still some states in which it is optimal to accept
only class-1 jobs, i.e., slower jobs, and some other states which admit only class-2 jobs according to the optimal admission policy. But this possibility also seems counterintuitive to us. We have numerically explored the optimal strategy in a large set of systems for which our results are inconclusive, i.e., when the parameters satisfy inequality (15):

**Example 4** We consider a system with 5 servers, for which we set \( \beta = 0 \), \( \mu_1 = 1 \) and \( r_2 = 1 \), and let \( r_1 \) to change between 1.1 to 5.1, \( \lambda_1 \) to change between 0.5 to 4, \( \rho_2 = \lambda_2/\mu_2 \) between 1 to 5, and \( \mu_2 \) between 1.1 to 5, all of them with the increments of 0.1. This way, a total of 2,420,640 examples is created, 108,300 of which satisfy inequality (15). In each of these 108,300 examples class 2 is preferred.

So the results of Example 4 provide a strong evidence that class 2 is always preferred in this “inconclusive” region.

The following example, which again satisfies inequality (15), shows more explicitly how the optimal policy changes for a two-server system:

**Example 5** Consider a system with 2 servers and parameters: \( \beta = 0 \), \( \lambda_1 = 0.4 \) and \( \lambda_2 = 2 \), \( \mu_1 = 0.05 \), \( \mu_2 = 1 \), and \( r_1 = 20 \) so that \( r_1\mu_1 = 1 \) and \( r_2\mu_2 = r_2 \). Let \( L_1 \) be the first and \( L_2 \) be the third term in (15), then we have \( L_1 = 1.5 \) and \( L_2 = 5.71 \). Now we let \( r_2 \), and so \( r_2\mu_2 / r_1\mu_1 \), vary in the range \([L_1, L_2]\): Figure 3 shows how the optimal policy changes for different values of \( r_2 \) in this range. Observe that class 2 is preferred in the whole inconclusive region.

Therefore, the following conjecture seems very plausible:

**Conjecture 2** There always exists a preferred class.

### 5.4 Sufficient Conditions for the \( r\mu \) Rule

From Theorems 1 and 2, we immediately obtain sufficient conditions for the \( r\mu \) rule to determine a preferred class.
Corollary 4 Under the following additional conditions, the $r\mu$ rule determines a preferred class:

(i) If $\frac{r_1\mu_1}{r_2\mu_2} \geq 1$, then class 1 is preferred.

(ii) If $\frac{r_2\mu_2}{r_1\mu_1} \geq \max \left\{ 1, \frac{\lambda_1(\mu_2 + \beta)}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)} \right\}$, then class 2 is preferred.

Proof. (i) If $r_1\mu_1 \geq r_2\mu_2$, then class 1 is preferred by Theorem 2 since

$$\frac{r_2\mu_2}{r_1\mu_1} \leq 1 < \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2}.$$ 

(ii) If $r_2\mu_2 \geq r_1\mu_1$ and:

$$\frac{r_2\mu_2}{r_1\mu_1} \geq \frac{\lambda_1(\mu_2 + \beta)}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)}$$

then class 2 is preferred by Theorem 1.

Note that this corollary gives the sufficient conditions. So, even if the given condition is not satisfied, the class with higher $r_i\mu_i$ might still be preferred.

6 GENERALIZATIONS AND FUTURE RESEARCH

We have also studied the system under batch arrivals (see Örmeci and Burnetas [11]) under the assumption that we can accept some of the jobs and reject the rest
from each batch. This is substantially different than the case in which we have to accept or reject the batch as a whole. For the latter case, an example is presented which violates any expected monotonicity properties of an optimal policy. Although some of the results can be replicated exactly for batch arrivals, they are not enough to characterize the optimal policy in terms of thresholds. However, we can still show the existence of a preferred class under certain conditions.

Here, we assume that jobs of each class bring fixed rewards. In Örmeci, Burnetas and Emmons [12], we have considered random rewards for each class. We have shown the existence of thresholds. Under random rewards, we cannot specify preferred classes, since the reward of each job, even if they are from the same class, varies. However, it is shown that there exist preferred jobs under certain conditions, where preferred jobs of a class are the jobs that bring at least a certain amount of reward.

Here we control the admission of jobs to the system directly. However, in some systems, it is more common to have an indirect control on the admission by using different prices, which may or may not depend on the state of the system. Miller and Buckman [9] have considered a static transfer pricing problem for one class in an $M/M/s/s$ system which serves as a model of a service department.

7 **APPENDIX**

7.1 **A Negative Lower Bound on $D_n(20)(x)$**

Throughout this section, we assume that $\frac{r_2 \mu_2}{r_1 \mu_1} \leq \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2}$, hence class-1 customers are preferred by Theorem 2. We first derive a new and negative lower bound on $D_n(20)(x)$, similar to the one in section 3.2.

Let $d_2$ be a lower bound on $D_n(20)(x)$ so that $d_2 \leq \min_{x \in S, n \geq 0}\{D_n(20)(x)\}$, and assume that $d_2 < 0$. We use the same sample path argument as in section 3.2 so that we have the following from inequality (5):

$$\lambda_1 \min\{R_2 - R_1, d_2\} + \lambda_2 d_2 + \mu_2 r_2 + (s - 1) \mu_2 d_2 \geq d_2,$$

where we have also used our assumption $d_2 < 0$. We may have two cases due to
the term with $\lambda_1$: If $d_2 < R_2 - R_1$, the maximal expression for $d_2$ will be $R_2$, thus positive. Therefore, we assume $d_2 \geq R_2 - R_1$ to have:

$$\lambda_1(R_2 - R_1) + \lambda_2d_2 + \mu_2r_2 + (s - 1)\mu_2d_2 \geq d_2.$$  

Now we define $\tilde{d}_2$ as:

$$\tilde{d}_2 = R_2 - \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta}R_1$$

using the uniformization, i.e., $\lambda_1 + \lambda_2 + s\mu_2 + \beta = 1$. Notice that $\tilde{d}_2 \geq R_2 - R_1$ as we have assumed. Moreover, it can be easily checked that:

$$\frac{r_2\mu_2}{r_1\mu_1} \leq \frac{\lambda_1(\mu_2 + \beta)}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)} \iff \tilde{d}_2 \leq 0.$$  

Now we can present this result in the following lemma without any proof, since the proof is very similar to that of Lemma 2, i.e., a combination of induction and the sample path argument we have just presented.

**Lemma 6**

If $\frac{r_2\mu_2}{r_1\mu_1} \leq \min \left\{ \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2}, \frac{\lambda_1(\mu_2 + \beta)}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)} \right\}$, then:

$$D_n(20)(x) \geq R_2 - \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta}R_1 \text{ for all } x \in S \text{ and for all } n \geq 0.$$  

**7.2 Proof of Lemma 5**

First, note that in section 4 we have seen that under both concavity conditions, class 1 is preferred.

The proof is by induction on the remaining number of transitions under either of the two conditions. Let $u_0$ be defined by (4). Then, the claim is true for $n = 0$ and we assume for $n$.

We define lower bounds $d_1^k$ and $d_2^k$ of $D_n(10)(x)$ and $D_n(12)(x)$, respectively, under concavity condition $k$ for all $x \in S$ and for all $n \geq 0$. Now we use the expressions of lower bounds given in Lemma 3.
Under concavity condition 1, the condition in part (i) of Lemma 3 holds, i.e.,
\[ \frac{r_2 \mu_2}{r_1 \mu_1} \leq \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta}, \]
so we can define:
\[ d_1^1 = \frac{r_1 \mu_1}{\lambda_1 + \mu_1 + \beta} \quad \text{and} \quad d_{12}^1 = \frac{r_1 \mu_1 (\lambda_1 + \mu_2 + \beta)}{(\mu_2 + \beta)(\lambda_1 + \mu_1 + \beta)} - \frac{r_2 \mu_2}{\mu_2 + \beta}. \]

Under concavity condition 2, we also have \( \frac{r_2 \mu_2}{r_1 \mu_1} \geq \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta} \) due to part (a) of concavity condition 2, so by part (ii) of Lemma 3:
\[ d_1^2 = \frac{r_1 \mu_1 (\lambda_2 + \mu_2 + \beta) - \lambda_2 r_2 \mu_2}{(\lambda_1 + \lambda_2 + \mu_2 + \beta)(\mu_1 + \beta) + \lambda_1 (\mu_2 - \mu_1)}, \quad \text{and} \quad d_{12}^2 = \frac{r_1 \mu_1 (\lambda_1 + \lambda_2 + \mu_2 + \beta) - r_2 \mu_2 (\lambda_1 + \lambda_2 + \mu_1 + \beta)}{(\lambda_1 + \lambda_2 + \mu_2 + \beta)(\mu_1 + \beta) + \lambda_1 (\mu_2 - \mu_1)}.
\]

We note that \(-d_{12}^k\) is an upper bound on \( D_n(21)(x) \) under concavity condition \( k \) for all \( x \in \mathcal{S} \) and for all \( n \geq 0 \), i.e., \(-d_{12}^k \geq \max_{\{x \in \mathcal{S}, n \geq 0\}} \{D_n(21)(x)\}\).

We also need a lower bound on \( D_n(20) \). Under both concavity conditions, the assumptions of Lemma 6 are satisfied: Consider concavity condition 1:
\[ \frac{r_2 \mu_2}{r_1 \mu_1} \leq \frac{\lambda_1}{\lambda_1 + \mu_1 + \beta} \leq \min \left\{ \frac{\lambda_2 + \mu_2 + \beta}{\lambda_2}, \frac{\lambda_1 (\mu_2 + \beta)}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)} \right\}, \]
where the first inequality is due to part (a) of concavity condition 1, and the second inequality is easy to check. For concavity condition 2, it has been shown in section 4 that \( \frac{r_2 \mu_2}{r_1 \mu_1} \leq \frac{\lambda_1 + \mu_2 + \beta}{\lambda_2} \) as a result of part (a) and (b), whereas we have \( \frac{r_2 \mu_2}{r_1 \mu_1} \leq \frac{\lambda_1 (\mu_2 + \beta)}{(\lambda_1 + \mu_2 + \beta)(\mu_1 + \beta)} \) directly from part (a). Thus, we can use the following \( d_2 \) as a lower bound for \( D_n(20)(x) \) by Lemma 6:
\[ d_2 = R_2 - \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} R_1. \]

Now we are ready to prove the statement for \( n + 1 \)-period systems. We have to show the following inequality:
\[ u_{n+1}(x + 2e_2) - u_{n+1}(x + e_2) \leq u_{n+1}(x + e_2) - u_{n+1}(x). \]

We will first show that this inequality is satisfied for \( v_n \)'s. We have several cases due to different optimal actions in states \((x + 2e_2)\) and \( x \): The first possible case is when the optimal actions in both states are the same. The second possible case arises
because of different actions in states \((x + 2e_2)\) and \(x\) due to a class-1 arrival. This can happen only when a class-1 job arrives at a system with all servers occupied in state \((x + 2e_2)\), since class 1 is preferred under both conditions. Finally, the optimal actions may be different upon a class-2 arrival: Note that the case \(a_n(x; 2) = 0\) and \(a_n(x + 2e_2; 2) = 1\) is not allowed due to the induction hypothesis, hence we can have only \(a_n(x; 2) = 1\) and \(a_n(x + 2e_2; 2) = 0\). Now we have a closer look at each of these cases:

**Case 1:** \(a(x + 2e_2; j) = a(x; j) = a\)

Then, \(v_n\)'s satisfy the inequality:

\[
v_n(x + 2e_2; j) - 2v_n(x + e_2; j) - v_n(x; j) \leq u_n(x + 2e_2 + ae_j) - 2u_n(x + e_2 + ae_j) - u_n(x + ae_j) \leq 0
\]

where the first inequality is due to the optimality of \(v_n\)'s, i.e., \(v_n(x; j) = \max\{u_n(x + e_j), u_n(x)\}\), and the second due to the induction hypothesis.

**Case 2:** \(a(x + 2e_2; 1) = 0\) and \(a(x; 1) = 1\)

Thus, we consider the statement for \(v_n(\cdot; 1)\) with \(x_1 + x_2 + 2 = s\):

\[
v_n(x + 2e_2; 1) - v_n(x + e_2; 1) = u_n(x + 2e_2) - u_n(x + e_1 + e_2)
\]

\[
v_n(x + e_2; 1) - v_n(x; 1) = u_n(x + e_1 + e_2) - u_n(x + e_1).
\]

It is difficult to relate the quantities (16) and (17) for general parameters. However, as we will show under both conditions \(d_2 \leq -d_{12}^k\), so that:

\[
u_n(x + 2e_2) - u_n(x + e_1 + e_2) \leq -d_{12}^k
\]

\[
\leq d_2 \leq u_n(x + e_1 + e_2) - u_n(x + e_1)
\]

by definition of \(-d_{12}^k\) and \(d_2\).

For concavity condition 1, we need to show that:

\[
-d_{12}^l = \frac{r_2 \mu_2}{\mu_2 + \beta} - \frac{r_1 \mu_1 (\lambda_1 + \mu_2 + \beta)}{(\mu_2 + \beta)(\lambda_1 + \mu_1 + \beta)}
\]

\[
\leq d_2 = \frac{r_2 \mu_2}{\mu_2 + \beta} - \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} \frac{r_1 \mu_1}{\mu_1 + \beta}.
\]
which can be verified to be equivalent to part (b) of concavity condition 1.

For concavity condition 2, we need to verify:

\[ -d_{12}^2 = \frac{r_2 \mu_2 (\lambda_1 + \lambda_2 + \mu_1 + \beta) - r_1 \mu_1 (\lambda_1 + \lambda_2 + \mu_2 + \beta)}{(\lambda_1 + \lambda_2 + \mu_2 + \beta)(\mu_1 + \beta) + \lambda_1 (\mu_2 - \mu_1)} \]

\[ \leq d_2 = \frac{r_2 \mu_2}{\mu_2 + \beta} - \frac{\lambda_1}{\lambda_1 + \mu_2 + \beta} \frac{r_1 \mu_1}{\mu_1 + \beta}. \]

With some algebraic manipulations, it can be shown, in combination with the right hand side of concavity condition 2 (a), that this inequality is equivalent to part (b) of concavity condition 2.

**Case 3:** \( a(x + 2e_2; 2) = 0 \) and \( a(x; 2) = 1 \)

\[
\begin{align*}
v_n(x + 2e_2; 2) - v_n(x + e_2; 2) - v_n(x + e_2; 2) + v_n(x; 2) & \leq \\
u_n(x + 2e_2) - u_n(x + e_2) - u_n(x + 2e_2) + u_n(x + e_2) & = 0
\end{align*}
\]

where the inequality follows from the optimality of \( v_n \)'s.

Now we can consider \( u_{n+1} \)'s:

\[
\begin{align*}
u_{n+1}(x + 2e_2) - 2u_{n+1}(x + e_2) + u_{n+1}(x) & = \\
\lambda_1[v_n(x + 2e_2; 1) - 2v_n(x + e_2; 1) + v_n(x; 1)] + \\
\lambda_2[v_n(x + 2e_2; 2) - 2v_n(x + e_2; 2) + v_n(x; 2)] + \\
x_1 \mu_1[u_n(x - e_1 + 2e_2) - 2u_n(x - e_1 + e_2) + u_n(x - e_1)] + \\
x_2 \mu_2[u_n(x + e_2) - 2u_n(x) + u_n(x - e_2)] + \\
\mu_2[u_n(x + e_2) - 2u_n(x) + u_n(x)] + \\
\mu_2[u_n(x + e_2) - 2u_n(x + e_2) + u_n(x)] + \\
[(s - x_1 - x_2 - 2) \mu_2 + x_1 (\mu_2 - \mu_1)][u_n(x + 2e_2) - 2u_n(x + e_2) + u_n(x)] & \geq 0
\end{align*}
\]

where the first two terms are non-positive since we have shown that \( v_n \)'s satisfy the given inequality, the third, fourth and seventh terms are non-positive by the induction hypothesis and fifth and sixth terms cancel out each other.

Thus the claim is true for all \( n \) under both conditions.
References


