KOÇ UNIVERSITY
MATH 101 - FINITE MATHEMATICS
Final Exam January 11, 2012
Duration of Exam: 110 minutes

INSTRUCTIONS: CALCULATORS ARE ALLOWED FOR THIS EXAM. No books, no notes, no questions and no talking allowed. You must always explain your answers and show your work to receive full credit. Use the back of these pages if necessary. Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.

Name:
Surname:
Signature:

Section (Check One):
- Section 1: E. Şüle Yavuz M-W-F (10:30)
- Section 2: E. Şüle Yavuz M-W-F (11:30)
- Section 3: Mehmet Sandereli M-W-F (13:30)
- Section 4: Ali Göktürk M-W-F (15:30)
- Section 5: Selda Kucukcligci T-Th-F (12:30)

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1. (15 points) Let \( \cos \left(4x - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} \). Find all solutions for \( x \) in the interval \([-2, 5]\).

\[
4x - \frac{\pi}{6} = A + 2\pi k
\]

\[
\cos A = -\frac{\sqrt{3}}{2} \Rightarrow \arccos A = \arccos \left(-\frac{\sqrt{3}}{2}\right)
\]

\[
A_1 = \frac{5\pi}{6} \quad A_2 = \frac{7\pi}{6}
\]

\[
4x - \frac{\pi}{6} = \frac{5\pi}{6} + 2\pi k
\]

\[
x_1 = \frac{\pi}{4}
\]

\[
4x - \frac{\pi}{6} = \frac{7\pi}{6} + 2\pi k
\]

\[
x_2 = \frac{3\pi}{4}
\]

\[
4x - \frac{\pi}{6} = \frac{9\pi}{6} + 2\pi k
\]

\[
x_3 = \frac{5\pi}{4}
\]

\[
4x - \frac{\pi}{6} = \frac{11\pi}{6} + 2\pi k
\]

\[
x_4 = \frac{7\pi}{4}
\]

\[
4x - \frac{\pi}{6} = \frac{13\pi}{6} + 2\pi k
\]

\[
x_5 = \frac{9\pi}{4}
\]

\[
4x - \frac{\pi}{6} = \frac{15\pi}{6} + 2\pi k
\]

\[
x_6 = \frac{11\pi}{4}
\]

\[
4x - \frac{\pi}{6} = \frac{17\pi}{6} + 2\pi k
\]

\[
x_7 = \frac{13\pi}{4}
\]

\[
4x - \frac{\pi}{6} = \frac{19\pi}{6} + 2\pi k
\]

\[
x_8 = \frac{15\pi}{4}
\]

We have 8 solutions for \( x \)!
2. (15 points) Solve the following system using Gauss Jordan Elimination method. Write the solution set and determine if the system is consistent, inconsistent, dependent or independent.

\[
\begin{align*}
\begin{cases}
    x_1 - 2x_2 + 2x_3 + 9x_5 &= 2 \\
x_1 - 2x_2 + 2x_3 + x_4 + 5x_5 &= 4 \\
3x_1 - 6x_2 + 3x_3 + 6x_4 + 21x_5 &= 3
\end{cases}
\end{align*}
\]

\[
\begin{bmatrix}
1 & -2 & 2 & 0 & 9 \\
1 & -2 & 1 & 5 & 4 \\
3 & -6 & 3 & 6 & 21
\end{bmatrix}
\xrightarrow{\text{for } \mathbf{r}_1\text{ and } \mathbf{r}_2\text{ to } \mathbf{r}_3}
\begin{bmatrix}
1 & -2 & 2 & 0 & 9 \\
0 & 0 & 4 & -4 & 2 \\
0 & 0 & 0 & -3 & 6
\end{bmatrix}
\xrightarrow{\text{row operations}}
\begin{bmatrix}
1 & -2 & 2 & 0 & 9 \\
0 & 0 & 4 & -4 & 2 \\
0 & 0 & 0 & -3 & 6
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -2 & 2 & 0 & 9 \\
0 & 0 & 4 & -4 & 2 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix}
\xrightarrow{\text{for } \mathbf{r}_2\text{ and } \mathbf{r}_1\text{ to } \mathbf{r}_1}
\begin{bmatrix}
1 & -2 & 2 & 0 & 9 \\
0 & 0 & 1 & -2 & 2 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -2 & 0 & 4 & 5 \\
0 & 0 & 1 & -2 & 2 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix}
\xrightarrow{\text{for } \mathbf{r}_3\text{ and } \mathbf{r}_2\text{ to } \mathbf{r}_2}
\begin{bmatrix}
1 & -2 & 0 & 4 & 5 \\
0 & 0 & 1 & -2 & 2 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & -2 & 2 \\
0 & 0 & 0 & 1 & -4
\end{bmatrix}
\xrightarrow{\text{for } \mathbf{r}_3\text{ to } \mathbf{r}_4\text{ and } \mathbf{r}_1\text{ to } \mathbf{r}_1}
\begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & -2 & 2 \\
0 & 0 & 0 & 0 & -6
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & -2 & 2 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 - 2x_2 + 21x_5 = -8 \\
x_3 - 5x_5 = 5 \\
x_4 - 4x_5 = 2
\end{bmatrix}
\]

\[
\begin{align*}
x_5 &= t \\
x_3 &= 5 + t \\
x_4 &= 2 + 4t \\
x_1 &= -8 - 21t + 2y
\end{align*}
\]

\[
\begin{align*}
solution set
\end{align*}
\]

\[
\begin{align*}
\{ -8 - 21t + 2y, y, 5 + t, 2 + 4t, t \}
\end{align*}
\]

dependent, consistent.
3. (15 points) Sketch the graph of the function

\[ f(x) = \begin{cases} 
\log(x - 2) - 1 & x > 2 \\
e^x & x \leq 2 
\end{cases} \]

by specifying at least 4 points on the graph. Find also \( x \) and \( y \)-intercepts.
A list of formulas:  \[ I = Prt; \quad A = P(1 + rt) \]
\[ A = P(1 + i)^n; \quad APY = (1 + \frac{i}{m})^m - 1; \quad A = Pe^{rt}; \quad APY = e^r - 1; \]
\[ FV = PMT \left( \frac{(1+i)^n-1}{i} \right); \quad PV = PMT \left( \frac{1-(1+i)^{-n}}{i} \right), \text{ where } i = \frac{r}{m} \text{ and } n = mt. \]

A: (20 points) Ahmet Uyank signed a 225,000 TL 25-year mortgage at 4.24% compounded monthly with Work Bank to buy his house 9 years ago. Whitebank now offers to refinance his mortgage at 9.9% compounded monthly. Ahmet Bey decides to accept the offer. His new 16-year mortgage will be in the amount of what he still owes Work Bank (after 9 years of making monthly payments to Work Bank).

(a) How much did Ahmet Bey pay Work Bank every month during the first nine years?

\[ \begin{align*}
225000 &= PMT \left( \frac{1-(1+\frac{0.124}{12})^{-12 \times 25}}{\frac{0.124}{12}} \right) \\
\Rightarrow PMT &= 2436.52
\end{align*} \]

(b) Show that the amount Ahmet Bey still owes Work Bank after nine years of monthly payments is 203,034.76 TL.

\[ \begin{align*}
PV_{16} &= 2436.52 \left( \frac{1- \left( 1 + \frac{0.124}{12} \right)^{-12 \times 16}}{\frac{0.124}{12}} \right) \\
PV_{16} &= 203,034.76 \quad \text{(proved)}
\end{align*} \]
(c) How much will Ahmet Bey have to pay Whitebank per month for the next 16 years?

\[ \text{new PV} = 203,034.76 \quad \rightarrow \quad \text{16 years.} \]

\[ 203,034.76 = \text{PMT} \times \left( \frac{1 - \left(1 + \frac{0.098}{12}\right)^{16 \times 12}}{0.098} \right) \]

\[ \text{new PMT} = 2,110.92 \]

(d) How much will Ahmet Bey save by refinancing his mortgage?

\[ \text{PMT}_1 \times (16 \times 12) = 467,811.84 = \text{He would have paid this much} \]

\[ (2,436.52) \]

\[ \text{PMT}_2 \times (16 \times 12) = 405,296.96 = \text{He paid this much} \]

\[ (2,110.92) \]

\[ 467,811.84 - 405,296.96 = 62,514.88 \]

[He saved this much]
5. (15 points) Maximize the function \( P = 10x_1 + 50x_2 + 10x_3 \) subject to the constraints

\[
\begin{align*}
x_1 + x_2 + x_3 & \leq 22 \\
3x_1 - x_2 + 2x_3 & \leq 24 \\
x_1 + x_2 + 3x_3 & \leq 36 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

\[
\begin{align*}
x_1 + x_2 + x_3 + s_1 & = 22 \\
3x_1 - x_2 + 2x_3 + s_2 & = 24 \\
x_1 + x_2 + 3x_3 + s_3 & = 36 \\
-10x_1 - 50x_2 - 10x_3 + P & = 0
\end{align*}
\]

\[
\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 22 \\
3 & -1 & 2 & 0 & 1 & 24 \\
1 & 1 & 3 & 0 & 0 & 36 \\
-10 & -50 & -10 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{align*}
x_1 & = 22 \\
x_2 & = -24 \\
x_3 & = 36
\end{align*}
\]

\[
\begin{align*}
x_1 + x_2 & \rightarrow x_2 \Rightarrow x_1 \\
21(50) + 84 & = 74 \\
21(-1) + 23 & = 23
\end{align*}
\]

\[
\begin{align*}
x_1 &= 0 \\
x_2 &= 22 \\
x_3 &= 0 \\
s_1 &= 46 \\
s_2 &= 14 \\
P &= 1100
\end{align*}
\]

Maximum value is 1100
6. (10 points) A textile company has 3 types of products type A, B and C. Each meter of type A is made from 2 kilograms of wool and 4 kilograms of cotton. Similarly each meter of type B is made from 3 kilograms of wool and 2 kilograms of cotton and type C is made from 5 kilograms of wool and 1 kilogram of cotton. In stock the company has 500 kilograms of wool and 300 kilograms of cotton. The machines can make at most 200 meters a day. The company makes a profit of 5 TL, 3 TL and 4 TL for each meter of type A, B and C, respectively. How many meter of each type should the company produce to maximize its daily profit?

Write the decision variables, appropriate equation(s) and inequalities so that this problem can be solved.

DO NOT SOLVE THE PROBLEM after you have written the equation(s) and inequalities.

\[
\begin{array}{c|c|c|c|c}
 & A & B & C & Total \\
\hline
W & 2 \text{ kg} & 3 \text{ kg} & 5 \text{ kg} & 500 \text{ kg} \\
\hline
\text{Cot} & 2 \text{ kg} & 2\text{ kg} & 1 \text{ kg} & 300 \text{ kg} \\
\hline
\end{array}
\]

\[x_1 \text{ meter } A, \ x_2 \text{ meter } B, \ x_3 \text{ meter } C, \ P= \text{profit}\]

Subject to constraints:
\[2x_1 + 3x_2 + 5x_3 \leq 500\]
\[4x_1 + 2x_2 + x_3 \leq 300\]
\[x_1 + x_2 + x_3 \leq 200\]
\[x_1, x_2, x_3 \geq 0\]

Maximize the objective function:
\[5x_1 + 3x_2 + 4x_3 = P\]
7. (15 points) Consider the objective function \( z = x_1 + 5x_2 \) subject to the constraints

\[
\begin{align*}
    x_1 + 4x_2 & \geq 30  \quad \text{(I)} \\
    3x_1 + x_2 & \geq 24  \quad \text{(II)} \\
    x_1, x_2 & \geq 0  \quad \text{(III)}
\end{align*}
\]

(a) Sketch the feasible region and find the corner points.

(b) If exist, find maximum and minimum values of \( z \) and where they occur.

\[
\begin{align*}
\text{A (0, 24)} & \quad \rightarrow 2 = x_1 + 5x_2 \quad \rightarrow 21 = 0 + 24 \cdot 5 = 120 \\
\text{B (6, 6)} & \quad \rightarrow 2 = x_1 + 5x_2 \quad \rightarrow 22 = 6.1 + 5.6 = 36 \\
\text{C (30, 0)} & \quad \rightarrow 2 = x_1 + 5x_2 \quad \rightarrow 23 = 30.1 + 5.0 = 35
\end{align*}
\]

no max value since it's unbounded.

min value is \( z_3 = 30 \)