MATH 102 FINAL EXAM SPRING 2008

Problem 1 (10 pts) Find the following limit.

\[ \lim_{n \to \infty} \frac{\ln(n + 1)}{\sqrt{n}} \]

Problem 2 (10 pts) Find the equation of the tangent line of the curve at \((0, 2)\).

\[ y = \frac{\ln(\cos x) + 2}{e^x} \]

Problem 3 (15 pts) Let \( f(x) = xe^x \n\)
a) Find all critical points, and intervals on which \( f \) is increasing & decreasing.
b) Find inflection points, and intervals on which \( f \) is concave up & concave down.
c) Find the asymptotes, if exist.
d) Sketch the graph of \( f \).

Problem 4 (15 pts) Compute the following improper integral.

\[ \int_1^\infty xe^{-x} \, dx \]

Problem 5 (10 pts) Find two numbers such that their difference is 18 and their product is minimum.

Problem 6 (15 pts) Compute the following integral.

\[ \int_0^1 \ln(x^2 + 1) \, dx \]

Problem 7 (10 pts) Find the area of the region between the curves \( y = \sqrt{8x} \) and \( y = x^2 \).

Problem 8 (15 pts) Find the volume of the solid obtained by rotating only one region between \( y = \sqrt{2 \sin(2x)} \) and \( y = 0 \) about \( x \)-axis.
1. \[ \lim_{n \to \infty} \frac{\ln(n+1)}{\sqrt{n}} = \lim_{n \to \infty} \frac{1}{n^{1/2}} = \lim_{n \to \infty} \frac{2\ln n}{n+1} = \lim_{n \to \infty} \frac{2\ln n}{n^{1/2}} = \lim_{n \to \infty} \frac{2x}{x} = \lim_{n \to \infty} \frac{1}{n} = 0 \]

2. \[ y = \frac{\ln(\cos x) + 2}{e^x} \quad y' = \frac{\frac{\sin x}{\cos x} \cdot e^x - (\ln(\cos x) + 2) \cdot e^x}{e^{2x}} \quad m = \frac{0.1 - 2.1}{1} = -2 \]

\[ \frac{y-2}{x-0} = -2 \quad y = -2x + 2 \]

3. a. \[ f'(x) = e^x + x \cdot e^x = (x+1)e^x \quad x = -1 \quad f'(x) \quad \text{local min.} \]

\[ f''(x) = e^x + (x+1)e^x = (x+2)e^x \quad f''(x) > 0 \quad (-1,\infty) \quad \text{concave up} \]

\[ f''(x) < 0 \quad (-\infty,-1) \quad \text{concave down} \]

b. \[ \lim_{x \to \infty} x \cdot e^x = \infty \quad \lim_{x \to -\infty} x \cdot e^x = \infty \quad \lim_{x \to 0} \frac{x}{e^x} = 0 \]

y = 0 horizontal asymptote.

c. \[ f(x) = e^x \quad f'(x) = e^x \quad f''(x) = e^x \]

\[ x = 1 \quad \text{inflexion pt.} \]

4. \[ \int_1^\infty xe^{-x} \, dx = \lim_{t \to \infty} \int_1^t xe^{-x} \, dx = \lim_{t \to \infty} \left[-(x+1)e^{-x}\right]_1^t = -e^{-t} - \left(-\frac{(t+1)}{e} + \frac{2}{e}\right) = \lim_{t \to \infty} \frac{-t+1}{e} + \frac{2}{e} + \frac{1}{e} = \frac{2}{e} \]

5. \[ x - y = 18 \quad \Rightarrow \quad y = x - 18 \]

\[ f(x) = x - y = x - (x - 18) = 18 \quad \Rightarrow \quad f'(x) = 2x - 18 \quad \Rightarrow \quad x = 9 \quad y = -9 \]
\[ \int_0^1 \ln(x+1) \, dx = x \ln(x+1) - \int_{x=0}^{x=1} \frac{2x \, dx}{x+1} = x \ln(x+1) - 2(x - \arctan(x)) \left|_0^1 \right. = (\ln 2 - 2(1 - \arctan 1)) - (0 - 0) \]
\[ = \ln 2 - 2 + \frac{\pi}{2} \]

7. \[ y = \sqrt{8x} \]
\[ x = \sqrt{8x} \]
\[ x^2 = 8x \]
\[ x^2 - 8x = 0 \]
\[ x(x - 8) = 0 \]
\[ x = 0 \]
\[ x = 8 \]

A = \int_0^2 \sqrt{8x} - x^2 \, dx = \left[ \frac{8}{3} x^{3/2} - \frac{x^3}{3} \right]_0^2 = \left[ \frac{16}{3} - \frac{8}{3} \right] = \frac{8}{3} \]

8. \[ y = \sqrt{2 \sin x} \]
\[ y = 0 \]
\[ \sqrt{2 \sin x} = 0 \]
\[ \sin x = 0 \]
\[ x = \pi \]
\[ y = \pi/\sqrt{2} \]

\[ \Phi = \pi \left( \frac{\pi}{\sqrt{2}} \sin x \right)^2 - \int_0^{\pi/2} 2 \sin 2x \, dx = \pi \left[ -\cos 2x \right]_0^{\pi/2} \]
\[ = \pi [1 - (-1)] = 2\pi \]