Problem 1 Find the following limits. Show all your work.

1a (6 pts) \[ \lim_{x \to \infty} \frac{5x^3 + 2x^2 - 1}{6x^3 + 4x} \]

1b (7 pts) \[ \lim_{x \to 0} \frac{x^2 - 3 \sin x}{2x + \cos x^2} \]

1c (7 pts) \[ \lim_{x \to \infty} x - \frac{1}{\sin \frac{1}{x}} \]

Problem 2 (15 pts) Two positive numbers are such that the sum of the first number and the square of the second number is 10. Find such numbers whose sum is the largest.

Problem 3 (25 pts) Let \( f(x) = x.(x - 3)^2 \)

a) Find all critical points, and intervals on which \( f \) is increasing & decreasing.

b) Find inflection points, and intervals on which \( f \) is concave up & concave down.

c) Find the asymptotes, if exist.

e) Sketch the graph of \( f \).

Problem 4 (20 pts) If \( f \) is differentiable everywhere, find \( a \) and \( b \).

\[
 f(x) = \begin{cases} 
 x^3 - ax^2 + b & x > 2 \\
 bx - 3a & x \leq 2 
\end{cases}
\]

Problem 5a (10 pts) Find the derivative of \( f(x) = \cos (\sqrt{x^{x+3}}) \)

5b (10 pts) Find the points on the curve \( x^2 + xy + y^2 = 3 \) where the tangent line is horizontal.

5c (5 pts) Find the points on the curve \( x^2 + xy + y^2 = 3 \) where the tangent line is vertical.
1. a. \( \lim_{x \to \infty} \frac{5x^2 + 2x^3 + 1}{6x^3 + 4x} = \lim_{x \to \infty} \frac{x^2 \left( 5 + \frac{2}{x} - \frac{1}{x^2} \right)}{x^3 \left( 6 + \frac{4}{x} \right)} = \lim_{x \to \infty} \frac{5x^2}{6x^2} = \frac{5}{6} \)

b. \( \lim_{x \to 0} \frac{x^2 - 3\sin x}{2x + \cos x^2} = \lim_{x \to 0} \frac{0 - 0}{0 + 1} = \lim_{x \to 0} \frac{0}{0} = 0 \)

c. \( \lim_{x \to \infty} \frac{\sqrt{x} - 1}{\sin \frac{1}{x}} = \lim_{h \to 0^+} \frac{\frac{\sin h - h}{h} \cdot \frac{1}{h}}{\frac{\sinh h}{h \cdot \sinh h}} = \lim_{h \to 0^+} \frac{\frac{\sin h}{h} \cdot \frac{1}{h}}{\frac{\sinh h}{h \cdot \sinh h}} = \lim_{h \to 0^+} \frac{\sin h}{h} = 1 \)

2. \( x + y = 10 \implies x = 10 - y \)

\( S = x + y \implies S(y) = 10 - y + y = 10 \) \( S'(y) = -2y + 2 \) \( S'(y) = 0 \implies y = 1 \)

\( \therefore x = 10 - \frac{3}{4} = \frac{37}{4} \)

3. \( f(x) = x \cdot (x - 3)^2 = x \cdot (x^2 - 6x + 9) = x^3 - 6x^2 + 9x \)

a. \( f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3) \) \( f' > 0 \) \( (-\infty, 1) \cup (3, \infty) \) \( \uparrow \)

b. \( f''(x) = 6x - 12 \) \( f''(x) = 6(x - 2) \) \( \uparrow \) \( \downarrow \) \( (2, \infty) \) \( (\infty, 2) \) \( \downarrow \)

c. \( \lim_{x \to \infty} f(x) = \pm \infty \) No hor. asymptote, No vert. asymptote
4. \( f(x) = \begin{cases} x^2 - ax^2 + b & x > 2 \\ bx - 2a & x \leq 2 \end{cases} \)

**Continuity at 2:** 
\[ 8 - 4a + b = 2b - 2a \implies a + b = 8 \]
\[ a = \frac{8}{2} \]
\[ b = \frac{20}{2} \]

**Diff. at 2:** 
\[ x = 2 \]
\[ 3x^2 - 2ax = b \]
\[ 12 - 4a = b \implies 4a + b = 12 \]

5. \( \cos\left(\sqrt{\frac{x}{x^2+1}}\right) = -\sin\left(\frac{\sqrt{x}}{x^2+1}\right) \cdot \left(\frac{\frac{1}{2\sqrt{x}} \cdot x^2 + 2x}{(x^2+1)^{\frac{3}{2}}}\right) \)

b. \( x^4 + xy + y^2 \geq 3 \)

**Horizontal tangent line:** 
\[ 2x + y + xy + 2yy' = 0 \]
\[ y' = 0 \implies 2x + y = 0 \]
\[ y = -2x \]
\[ y = -2 \]
\[ x = \pm 1 \]
\[ (1, -2) (-1, 2) \]

**Vertical tangent line:** 
\( y' = \infty \) (or \( x = 0 \))

\[ y' = -\frac{(2x+y)}{x+ly} \] 
\[ x + ly = 0 \]
\[ x = -ly \]

\[ x^2 + x + (2x)^2 \geq 1 \]

\[ y = 2 \]
\[ y = 1 \]
\[ y = \pm 1 \implies x = \pm 2 \]