Problem 1  (15 points)

Find the area between the curves $y = x^2$ and $x = y^2$.

\[ y = x^2 = y^4 \Rightarrow y^4 - y^2 = 0 \Rightarrow y^2(y^2 - 1) = 0 \Rightarrow y = 0 \text{ or } y = 1. \]

Therefore the points of intersection of the given curves are $(0,0)$ and $(1,1)$. If we denote the area between the curves as $A$, then $A = \int_0^1 \left( \sqrt{y} - y^2 \right) \, dy$.

\[ \int_0^1 x^2 \, dx = \frac{2}{3} \left[ \frac{x^3}{3} \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}. \]

\[ +7 \text{ points} \]

\[ +3 \text{ points} \]

\[ +5 \text{ points} \]
Problem 2 (15 points)

Calculate the following integrals. Show all your work.

(a) (7 points) $\int_1^e \frac{\sin(\ln x)}{x} \, dx$.

\[ \int_1^e \frac{\sin(\ln x)}{x} \, dx = \int_1^e \frac{\sin(u)}{e^u} \, du \quad \text{where} \quad u = \ln x \]

\[ = \sin(u) \bigg|_1^e - \int_1^e \cos(u) \, du \bigg|_1^e \]

\[ = \sin(e) - \cos(e) - (-\sin(1) + \cos(1)) \]

(b) (8 points) Evaluate $\int_1^4 e^{\sqrt{x}} \, dx$.

\[ \int_1^4 e^{\sqrt{x}} \, dx = \int_1^4 e^{u} \, du \quad \text{where} \quad u = \sqrt{x} \]

\[ = \frac{e^u}{2} \bigg|_1^4 = \frac{e^{\sqrt{4}}}{2} - \frac{e^{\sqrt{1}}}{2} = \frac{e^2}{2} - \frac{e}{2} \]

We conclude that $\int_1^4 e^{\sqrt{x}} \, dx = \frac{e^2}{2} - e$. 

+4 points

+3 points

+3 points

+2 points

+1 points

+1 points
Problem 3 (10 points)

Use calculus to find two non-negative numbers such that their sum is 1 and sum of their squares is as small as possible.

\[ x+y \geq 0 \text{ such that } x+y=1. \text{ We want to minimize the function } f \text{ defined as } f(x,y) = x^2 + y^2. \] Due to the condition \( x+y=1 \), we can express \( f \) as a function of \( x \) only as

\[ f(x) = x^2 + (1-x)^2 = 2x^2 - 2x + 1, \] which is valid for \( x \geq 0 \).

2 points

Note that we're searching for the absolute minimum of \( f \) on the closed interval \([0,1]\).

Since \( f \) is continuous on that interval, we need to check only the critical points and the endpoints of the given interval: \( f \) is a polynomial hence differentiable on \([0,1]\).

Differentiating we find \( f'(x) = 4x - 2 \). As a result, the critical point of \( f \) is \( (0.5, 0.5) \) where \( f'(x) = 0 \). Solving for \( x \) we find \( 4x - 2 = 0 \) \( \Rightarrow x = \frac{1}{2} \) hence \( y = 2\cdot \frac{1}{2}^2 - 2\cdot \frac{1}{2} + 1 = \frac{1}{2} \). The critical point of \( f \) is \( (0.5, 0.5) \).

3 points

Checking the endpoints we get \((0,1)\) and \((1,0)\). Among these three points, \((0.5, 0.5)\) is the one with the smallest value for \( f \) hence we conclude that the absolute minimum value of \( f \) is \( \frac{1}{2} \), taken at the point \( x = \frac{1}{2} \).

2 points

As a conclusion, \( x = y = \frac{1}{2} \) are those numbers that we're looking for.
Problem 4 (15 points)
Use calculus to prove that the line $y = 4x + 7$ intersects the curve $y = 3 \sin x$ at exactly one point.

Let us define a function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = 3 \sin x - 4x - 7$. Our aim is to prove that this function has exactly one root. 

First of all, let us show that $f$ has at least one root. $f$ is continuous on $\mathbb{R}$ hence on any interval, the hypotheses of the intermediate value theorem hold. 

For $x = -\frac{3\pi}{2}$, $f(-\frac{3\pi}{2}) = 3 \sin (-\frac{3\pi}{2}) - 4(-\frac{3\pi}{2}) - 7 = 3 + 6\pi - 7 = 6\pi - 4 > 0$. For $x = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = 3 \sin (\frac{\pi}{2}) - 4(\frac{\pi}{2}) - 7 = 3 - 2\pi - 7 = -2\pi - 4 < 0$. As a result of the intermediate value theorem, we conclude that there exists $c \in (-\frac{3\pi}{2}, \frac{\pi}{2})$ with $f(c) = 0$.

Then we should show that $f$ has at most one root. Assume that $f$ has two distinct roots $x_1$ and $x_2$ with $x_1 < x_2$. $f$ is differentiable on $\mathbb{R}$ hence it satisfies all the necessary hypotheses for Rolle's theorem (or the mean value theorem) on $[x_1, x_2]$. By Rolle's theorem, then, we conclude that there exists $x_0 \in (x_1, x_2)$ with $f'(x_0) = 0$. By differentiating $f$ and putting $x_0$ in place of $x$ we conclude that $f'(x) = 3 \cos x - 4 \Rightarrow f'(x_0) = 3 \cos x_0 - 4$ and $f'(x_0) = 0 \Rightarrow 3 \cos x_0 - 4 = 0 \Rightarrow \cos x_0 = \frac{4}{3} > 1$ which is impossible.

Thus our assumption that “$f$ has two distinct roots” should be false. This observation finishes our work.
Problem 5 (20 points)

(a) (13 points) Determine the radius and the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(5x-3)^n}{n} \).

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(5x-3)^{n+1}}{n+1} \frac{n}{(5x-3)^n} \right| = \frac{|5x-3|}{1} \]

By the ratio test, \( \sum_{n=1}^{\infty} a_n \) converges absolutely for \( |5x-3| < 1 \) and diverges for \( |5x-3| > 1 \).

Rearranging \( |5x-3| < 1 \) as \( |x - \frac{3}{5}| < \frac{1}{5} \), we find the radius of convergence as \( R = \frac{1}{5} \). We're sure about the convergence of the series \( \sum_{n=1}^{\infty} \frac{(5x-3)^n}{n} \) on \( (-\frac{2}{5}, \frac{4}{5}) \). In order to determine the interval of convergence, we need to check the behavior of the numerical series resulting from putting \( x = \frac{2}{5} \) and \( x = \frac{4}{5} \) in the given series. For \( x = \frac{2}{5} \) we obtain the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) which is convergent by the alternating series test. For \( x = \frac{4}{5} \) we obtain the famous harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) which is divergent by the p-test. As a result, the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(5x-3)^n}{n} \) is \( [-\frac{2}{5}, \frac{4}{5}] \).

(b) (7 points) Represent the function \( f(x) = \frac{1}{3x+4} \) as a power series.

\[ f(x) = \frac{1}{3x+4} = \frac{1}{4} \cdot \frac{1}{1 - \left( -\frac{3x}{4} \right)} = \frac{1}{4} \sum_{n=0}^{\infty} \left( -\frac{3x}{4} \right)^n \text{ for } \left| -\frac{3x}{4} \right| < 1 \]

or \( f(x) = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{4^{n+1}} \) valid for \( 1x < \frac{4}{3} \).

- Any attempt to use \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \longrightarrow +5 \text{ points} \)

- Correct Taylor series formula for a general \( f \) \( \longrightarrow +4 \text{ points} \)
Problem 6 (20 points)
Suppose that the series $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -5$ and diverges when $x = 10$. For each of the following series, determine whether it is convergent or divergent; you need to justify your answer for full credit. (If a definite answer cannot be determined from the given information, give examples to explain why.)

(a) (4 points) $\sum_{n=0}^{\infty} c_n x^n$

$\sum_{n=0}^{\infty} c_n x^n$ converges for $|x|<R$ where $R$ is the radius of convergence. Since $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -5$, $R \geq 5$. Hence $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 1$. Thus $\sum_{n=0}^{\infty} c_n$ converges.

(b) (4 points) $\sum_{n=0}^{\infty} (-1)^n c_n 11^n$

Since $\sum_{n=0}^{\infty} c_n x^n$ diverges when $x = 10$, $R < 10$. Hence for $x = -11$, $|x|>R$ and therefore $\sum_{n=0}^{\infty} (-1)^n c_n 11^n$ diverges.
not, since $R > 7$ or $R > 7$ both may happen. We cannot decide whether $\sum_{n=0}^{\infty} c_n z^n$ converges.

We have $10 > R \geq 5$. From this information,

\[ \sum_{n=0}^{\infty} c_n (-4)^n \]

converges.

Since $R \geq 5$, $|z| = 4 > R$. Hence

\[ \sum_{n=0}^{\infty} c_n (-4)^n \]
Problem 7  (15 points)

Determine whether the series below are convergent or divergent. Justify your answer by explicitly stating what test you are appealing to and how you use that test.

(a)  (5 points)  \[ \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \] Since \( \frac{1}{n(\ln n)^2} > 0 \) for every \( n \geq 2 \)

and \( f(x) = \frac{1}{x(\ln x)^2} \) is continuous, decreasing for all \( x \geq N \)

we can apply integral test. \( \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \int_{2}^{\infty} \frac{du}{u^2} = -\frac{1}{u} \)

Hence \( \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \lim_{b \to \infty} \left( \frac{1}{\ln 2} - \frac{1}{\ln b} \right) = \frac{1}{\ln 2} \)

Thus \( \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx \) converges, so \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) converges.

(b)  (5 points)  \[ \sum_{n=1}^{\infty} \frac{1}{1 + \sin(1/n)} \]

Since \( \lim_{n \to \infty} \frac{1}{1 + \sin(1/n)} = \lim_{n \to \infty} \frac{1}{1 + \sin(1/n)} = 1 \neq 0 \), by n-th term test,

\[ \sum_{n=1}^{\infty} \frac{1}{1 + \sin(1/n)} \] diverges.

(c)  (5 points)  \[ \sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1} \]

Since \( \frac{n^2}{n^4 - 1} < \frac{n^2 + 1}{n^4 - 1} \), we have

\[ \sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1} < \sum_{n=2}^{\infty} \frac{n^2 + 1}{n^4 - 1} = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \] Now \( \frac{1}{n^2 - 1} = \frac{1}{(n-1)(n+1)} = \frac{1}{2} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \)

Hence \( \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{1}{2} \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{1}{2} \lim_{N \to \infty} \sum_{n=2}^{N} \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \)

\[ = \frac{1}{2} \lim_{N \to \infty} \left( 1 + \frac{1}{2} - \frac{1}{N+1} \right) = \frac{1}{2} \left( 1 + \frac{1}{2} \right) = \frac{3}{4} \], Hence \( \sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1} \) is convergent and by comparison test \( \sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1} \) is convergent.