KOÇ UNIVERSITY
MATH 106
FIRST EXAM MARCH 26, 2012
Duration of Exam: 90 minutes

INSTRUCTIONS: No calculators may be used on the test. No questions, and talking allowed. You must always explain your answers and show your work to receive full credit. Use the back of these pages if necessary. Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

Solutions by Ali Alp Uzman and Candan Güdüçü

(Check One): (Emre Alkan) : —— (Burak Özbağcı): ——

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>POINTS</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td></td>
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<tr>
<td>7</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>100</td>
<td></td>
</tr>
</tbody>
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Problem 1a (5 pts) Find the limit
\[ \lim_{x \to 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6}. \]

**ANSWER:**
Notice that
\[ \lim_{x \to 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6} = \lim_{x \to 2^+} \frac{(x + 2)(x - 4)}{(x - 2)(x - 3)} = +\infty \]
since the numerator approaches to \(-8\) and the denominator approaches to 0 with **negative** values. Hence this limit does not exist.

Problem 1b (5 pts) Find the limit
\[ \lim_{t \to 0} \left( \frac{1}{t \sqrt{1 + t}} - \frac{1}{t} \right). \]

**ANSWER:**
\[
\begin{align*}
\lim_{t \to 0} \left( \frac{1}{t \sqrt{1 + t}} - \frac{1}{t} \right) &= \lim_{t \to 0} \left( \frac{1}{t \sqrt{1 + t}} - \frac{\sqrt{1 + t}}{t \sqrt{1 + t}} \right) \\
&= \lim_{t \to 0} \left( \frac{1 - \sqrt{1 + t}}{t \sqrt{1 + t}} \right) \\
&= \lim_{t \to 0} \left( \frac{1 - \sqrt{1 + t}}{t \sqrt{1 + t}} \right) \left( \frac{1 + \sqrt{1 + t}}{1 + \sqrt{1 + t}} \right) \\
&= \lim_{t \to 0} \left( \frac{-t}{t \sqrt{1 + t}(1 + \sqrt{1 + t})} \right) \\
&= -\frac{1}{2}
\end{align*}
\]

Problem 1c (5 pts) Find the limit, if it exists. If the limit does not exist, explain why.
\[ \lim_{x \to -6} \frac{2x + 12}{|x + 6|} \]

**ANSWER:**
Since \( \lim_{x \to -6^+} \frac{2x + 12}{|x + 6|} = 2 \neq -2 = \lim_{x \to -6^-} \frac{2x + 12}{|x + 6|} \), we conclude that \( \lim_{x \to -6} \frac{2x + 12}{|x + 6|} \) does not exist.

Problem 1d (5 pts) Using the Squeeze theorem, find the limit
\[ \lim_{x \to 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x}. \]
ANSWER:

Observe that $-1 \leq \sin \frac{\pi}{x} \leq 1$ for all $x \in \mathbb{R}$. We may multiply each side with $\sqrt{x^3 + x^2}$, since $\sqrt{x^3 + x^2} \neq 0$ when $x$ is near 0. Then

$$-\sqrt{x^3 + x^2} \leq \sqrt{x^3 + x^2} \cdot \sin \frac{\pi}{x} \leq \sqrt{x^3 + x^2}.$$

Notice that

$$\lim_{x \to 0} (-\sqrt{x^3 + x^2}) = 0 = \lim_{x \to 0} \sqrt{x^3 + x^2}.$$

Then by Squeeze Theorem,

$$\lim_{x \to 0} \sqrt{x^3 + x^2} \cdot \sin \frac{\pi}{x} = 0.$$

Problem 2 (10 pts) Find values of $a$ and $b$ such that the function

$$f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2} & \text{if } x < 2 \\
ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\
2x - a + b & \text{if } x \geq 3
\end{cases}$$

is continuous everywhere.

ANSWER:

The function $f(x)$ is defined as a continuous functions for all $x \in \mathbb{R} \setminus \{2, 3\}$, so we only need to make sure the function is continuous at the points $x = 2$ and $x = 3$.

\begin{itemize}
    \item $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$
    
    \[ \Rightarrow \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^+} (ax^2 - bx + 3) \Rightarrow 4 = 4a - 2b + 3 \]

    \item $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x)$
    
    \[ \Rightarrow \lim_{x \to 3^-} (ax^2 - bx + 3) = \lim_{x \to 3^+} (2x - a + b) \Rightarrow 9a - 3b + 3 = 6 - a + b \]

\end{itemize}

$4a - 2b + 3 = 4 \Rightarrow 4b = 8a - 2$. Also, $9a - 3b + 3 = 6 - a + b \Rightarrow 10a - 4b = 3$. Then $10a - (8a - 2) = 3 \Rightarrow a = 1/2 \Rightarrow b = 1/2$.

To sum up, for $a = b = 1/2$, the given function $f(x)$ is continuous everywhere.
Problem 3 (10 pts) Is there a number that is exactly 1 more than its cube? Justify your answer.

ANSWER:
We need to find a real number $x$ such that $x = x^3 + 1 \Rightarrow x^3 - x + 1 = 0$. Let $f(x) := x^3 - x + 1$. Observe that $f(-2) = -5$, while $f(-1) = 1$. So $f(-2) = -5 < 0 < 1 = f(-1)$. Then by Intermediate Value Theorem, there is $c \in (-2, -1)$ such that $f(c) = c^3 - c + 1 = 0$, and hence $c = c^3 + 1$.

Problem 4 (15 pts) Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x^2 - x}{x^2 - 6x + 5}.$$ 

ANSWER:
\[
\begin{align*}
\lim_{x \to \infty} \frac{x^2 - x}{x^2 - 6x + 5} &= \lim_{x \to \infty} \frac{x^2(1 - \frac{1}{x})}{x^2(1 - \frac{6}{x} + \frac{5}{x^2})} = 1 \\
\lim_{x \to -\infty} \frac{x^2 - x}{x^2 - 6x + 5} &= \lim_{x \to -\infty} \frac{x^2(1 - \frac{1}{x})}{x^2(1 - \frac{6}{x} + \frac{5}{x^2})} = 1 \\

\text{Thus } y = 1 \text{ is the only horizontal asymptote of the curve.}
\end{align*}
\]
\[
\begin{align*}
\lim_{x \to 5^-} \frac{x^2 - x}{x^2 - 6x + 5} &= \lim_{x \to 5^-} \frac{x(x - 1)}{(x - 1)(x - 5)} = \lim_{x \to 5^-} \frac{x}{x - 5} = -\infty \\
\lim_{x \to 5^+} \frac{x^2 - x}{x^2 - 6x + 5} &= \lim_{x \to 5^+} \frac{x(x - 1)}{(x - 1)(x - 5)} = \lim_{x \to 5^+} \frac{x}{x - 5} = +\infty \\

\text{Thus } x = 5 \text{ is the only vertical asymptote of the curve.}
\end{align*}
\]
\[
\begin{align*}
\lim_{x \to 1^-} \frac{x^2 - x}{x^2 - 6x + 5} &= \lim_{x \to 1^-} \frac{x(x - 1)}{(x - 1)(x - 5)} = \lim_{x \to 1^-} \frac{x}{x - 5} = \frac{-1}{4} \\
\lim_{x \to 1^+} \frac{x^2 - x}{x^2 - 6x + 5} &= \lim_{x \to 1^+} \frac{x(x - 1)}{(x - 1)(x - 5)} = \lim_{x \to 1^+} \frac{x}{x - 5} = \frac{-1}{4} \\

x=1 \text{ isn't a vertical asymptote of the curve.}
\end{align*}
\]
Problem 5 (15 pts) Find an equation of the tangent line to the curve $y = x \sqrt{x}$ that is parallel to the line $y = 3x + 1$.

ANSWER:
Let $(a, a\sqrt{a})$ be a point on the graph of $y = x \sqrt{x}$ where the slope of the tangent line is 3. The only solution to the equation $\frac{3\sqrt{a}}{2} = 3$ is $a = 4$, and thus $y = a\sqrt{a} = 8$. Therefore the tangent line at $(4, 8)$ is given by $y - 8 = 3(x - 4)$ or equivalently $y = 3x - 4$.

Problem 6 (15 pts) Find $\frac{dy}{dx}$ by implicit differentiation from the equation $e^y \cos x = 1 + \sin(xy)$.

ANSWER:

$$
\frac{d(e^y \cos x)}{dx} = \frac{d(1 + \sin(xy))}{dx}
$$

$$
e^y \cos x \frac{dy}{dx} + e^y (-\sin x) = y \cos(xy) + x \cos(xy) \frac{dy}{dx}
$$

$$
(e^y \cos x - x \cos(xy)) \frac{dy}{dx} = y \cos(xy) + e^y \sin x
$$

$$
\frac{dy}{dx} = \frac{y \cos(xy) + e^y \sin x}{e^y \cos x - x \cos(xy)}
$$

Problem 7 (15 pts) Two sides of a triangle have lengths 12m. and 15m. The angle between them is increasing at a rate 2 degrees per minute. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60 degrees.

ANSWER:
Let $\theta(t)$ denote the angle depending on time $t$, between two sides of fixed length. Also we denote the length of the third side with $c(t)$ which is time dependent. We are given the rate of change of $\theta(t)$ with respect to time, that is,

$$
\frac{d}{dt} \theta(t) = 2 \text{ degree/minute} = \frac{\pi}{90} \text{ radian/minute}
$$
Also, we know that \((c(t))^2 = a^2 + b^2 - 2ab \cos(\theta(t))\) from the Law of Cosines. We may substitute \(a, b\) with their respective values, since they both are constants.

\[
(c(t))^2 = 369 - 360 \cos(\theta(t))
\]

Furthermore, by taking the square roots of each side, we obtain the following:

\[
c(t) = \sqrt{369 - 360 \cos(\theta(t))}
\]

Notice that since \(c\) is length, it can’t have any negative values. Then by using the Chain Rule:

\[
\frac{dc}{dt} = \left(\frac{360 \sin \theta(t)}{2\sqrt{369 - 360 \cos(\theta(t))}}\right) \cdot \left(\frac{d\theta(t)}{dt}\right)
\]

and hence

\[
\frac{dc}{dt}\bigg|_{\theta=\pi/3} = \frac{\pi}{3\sqrt{7}} \left(\frac{m.}{min}\right).
\]