KOÇ UNIVERSITY
MATH 106
SECOND EXAM MAY 7, 2012

Duration of Exam: 105 minutes

INSTRUCTIONS: No calculators may be used on the test. No questions, and talking allowed. You must always explain your answers and show your work to receive full credit. Use the back of these pages if necessary. Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

Solutions by Ali Alp Uzman and Candan Güdücü

(Check One): (Emre Alkan): ——
(Burak Özbağcı): ——

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Problem 1 (10 pts) Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all $x$? Justify your answer.

**SOLUTION:**

**Claim.** There is no differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all $x \in \mathbb{R}$.

**Proof.** Suppose that there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$, for all $x \in \mathbb{R}$. Then $f$ is differentiable on $(0, 2)$ and continuous on $[0, 2]$. So, by the Mean Value Theorem there exists a number $c \in (0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{5}{2} > 2$$

This contradicts to the assumption that $f'(x) \leq 2$ for all $x \in \mathbb{R}$ which proves our claim.

Problem 2 (15 pts) Find $\lim_{x \to \infty} (e^x + x)^{1/x}$ if it exists.

**SOLUTION:**

Let $f(x) = (e^x + x)^{1/x} \Rightarrow \ln(f(x)) = \frac{1}{x} \cdot \ln(e^x + x) = \frac{\ln(e^x + x)}{x}$

$\Rightarrow \lim_{x \to \infty} \ln(f(x)) = \lim_{x \to \infty} \frac{\ln(e^x + x)}{x}$. This is the indeterminate form $\frac{\infty}{\infty}$. We apply L’Hospital’s Rule to calculate

$$\lim_{x \to \infty} \frac{\ln(e^x + x)}{x} = \lim_{x \to \infty} \frac{e^x + 1}{e^x + x} = \lim_{x \to \infty} \frac{e^x}{e^x + 1} = 1.$$

Then $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{\ln(f(x))} = e^1 = e.$
Problem 3 (15 pts) Sketch the graph of a twice differentiable function \( f : \mathbb{R} \rightarrow \mathbb{R} \) that satisfies all of the following conditions:

\[
\begin{align*}
 f(0) &= 2, \quad f(2) = -1, \quad f(4) = 1, \\
 f'(0) &= f'(2) = f'(4) = 0, \\
 f'(x) &> 0 \text{ if } x < 0 \text{ or } 2 < x < 4, \\
 f'(x) &< 0 \text{ if } 0 < x < 2 \text{ or } x > 4, \\
 f''(x) &> 0 \text{ if } 1 < x < 3, \quad f''(x) < 0 \text{ if } x < 1 \text{ or } x > 3.
\end{align*}
\]

SOLUTION:

• \( f(0) = 2, \quad f(2) = -1, \quad f(4) = 1 \Rightarrow \) The points \((0, 2), (2, -1), (4, 1)\) are on the graph of \( f \).

• \( f'(0) = f'(2) = f'(4) = 0 \Rightarrow \) There are extrema at \( x=0,2,4 \). \( f'(x) > 0 \) if \( x < 0 \) or \( 2 < x < 4 \Rightarrow f(x) \) is increasing for all \( x \in (-\infty, 0) \cup (2, 4) \).

• \( f'(x) < 0 \) if \( 0 < x < 2 \) or \( x > 4 \Rightarrow f(x) \) is decreasing for all \( x \in (0, 2) \cup (4, \infty) \).

• \( f''(x) > 0 \) if \( 1 < x < 3 \Rightarrow f(x) \) is concave upward for all \( x \in (1, 3) \).

• \( f''(x) < 0 \) if \( x < 1 \) or \( x > 3 \Rightarrow f(x) \) is concave downward for all \( x \in (-\infty, 1) \cup (3, \infty) \).

• Also the last two informations imply that \((1, f(1))\) and \((3, f(3))\) are inflection points for \( f(x) \).
Problem 4 (15 pts) **Sketch the region** enclosed by the curves \( y = \cos(\pi x) \) and \( y = 4x^2 - 1 \), and find its area.

**SOLUTION:**

In the figure above, the red curve is the graph of \( y = \cos(\pi x) \), while the magenta curve is the graph of \( y = 4x^2 - 1 \). They intersect with each other at \((-1/2, 0)\) and \((1/2, 0)\), hence the integral giving the area of the region is as follows:

\[
A = \int_{x=-1/2}^{1/2} (\cos(\pi x) - (4x^2 - 1))\,dx = 2 \int_{x=0}^{1/2} (\cos(\pi x) - (4x^2 - 1))\,dx
\]

This is true because \((\cos(\pi x) - (4x^2 - 1))\) is an even function. It follows that

\[
A = 2 \left( \frac{\sin(\pi x)}{\pi} - \frac{4x^3}{3} + x \right) \bigg|_{x=0}^{1/2} = \frac{2}{3} + \frac{2}{\pi}.
\]
Problem 5 (15 pts) Find the volume of the solid obtained by rotating the region bounded by \( y = x^2 \) and \( x = y^2 \) about the line \( y = 1 \).

**SOLUTION:**

\[
\text{volume} = \int_0^1 \pi (1 - x^2)^2 dx - \int_0^1 \pi (1 - \sqrt{x})^2 dx
\]

\[
= \int_0^1 \pi (1 - 2x^2 + x^4) dx - \int_0^1 \pi (1 - 2\sqrt{x} + x) dx
\]

\[
= \pi \left[ x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1 - \pi \left[ x - \frac{4x\sqrt{x}}{3} + \frac{x^2}{2} \right]_0^1
\]

\[
= \pi \left( 1 - \frac{2}{3} + \frac{1}{5} - (0 - 0 + 0) \right) - \pi \left( 1 - \frac{4}{3} + \frac{1}{2} - (0 - 0 + 0) \right)
\]

\[
= \frac{11\pi}{30}
\]
**Problem 6 (15 pts)** A box with a square base and open top must have a volume of 32,000 cm$^3$. Find the dimensions of the box that minimize the amount of material used.

**SOLUTION:**

Edge of the square base=$x$ and height of the box=$h$

$$Volume = x^2 . h = 32000cm^3 \rightarrow h = \frac{32000}{x^2}$$

$$Area(x, h) = x^2 + 4xh \rightarrow A(x) = Area(x) = x^2 + \frac{128000}{x}$$

To minimize area we need to find absolute minimum of $A(x)$ on $(0, \infty)$:

$$A'(x) = 2x - \frac{128000}{x^2} = \frac{2x^3 - 128000}{x^2} = 0$$

Hence, $x = 40$ is the only critical point of the area function $A(x)$ on $(0, \infty)$. One checks that $A''(40) > 0$ to verify that $x = 40$ is a minimizer of $A(x)$. It follows that the dimensions of the box that minimize the amount of material used are $x = 40cm$ and $h = \frac{32000}{x^2} = 20cm$.

**Problem 7 (15 pts)** Find the interval on which $f(x) = \int_0^x (1-t^2)e^{t^2} dt$ is increasing.

**SOLUTION:**

Since $(1-t^2)e^{t^2}$ is a continuous function consisting of polynomials and exponentials, by The Fundamental Theorem of Calculus, we have $f'(x) = (1-x^2)e^{x^2}$. We conclude that $f$ is increasing on $(-1, 1)$ since this is the interval on which $f'(x) > 0$. (Notice that $e^{x^2}$ is always positive.)