MODELING OPTIMIZATION PROBLEMS

INDR 262-INTRODUCTION TO OPTIMIZATION METHODS

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DECISION MAKING

Structuring the problem

- Define the problem
- Identify the alternatives
- Determine the criteria

Analyzing the problem

- Quantitative Analysis
- Qualitative Analysis

  ▪ Summary and Evaluation
  ▪ Make the decision

Determine the criteria
Quantitative Analysis Process

1. Model development
2. Data preparation
3. Model solution
4. Analysis of the solution and report generation

Potential reasons for a quantitative analysis approach to decision making

- The problem is complex, has significant impact, is large-scale or repetitive.
CONCEPT OF A MODEL

- Models are representations of real objects or systems.
- Building a model helps understanding a system.
- Generally, experimenting with models (compared to experimenting with the real system) requires less time, is less expensive, involves less risk.
Mathematical models represent real world problems through a system of mathematical relationships (formulas and expressions) based on key assumptions, estimates, or statistical analyses.

Examples of mathematical models:
- Simulation models, econometric models, time series models, mathematical programming models.
Relate **decision variables** with input parameters.

Maximize or minimize some **objective function** subject to **constraints**.

- **Objectives** (minimize risk, maximize profit, etc.)
- **Constraints** (capacities, budget limits, etc.)
Defining the Problem

- Study the relevant system and develop a well-defined statement of the problem
  - Objectives
  - Constraints
  - Interrelationships
  - Alternatives
  - Time Limits
 Decision Variables: variable values to be determined.

Objective Function: measure of performance

Constraints: any restrictions on the values that can be assigned to decision variables.

Parameters: the constants in the constraints and the objective function.
GUIDELINES FOR MODEL DEVELOPMENT

1. Write a verbal description of the objective and each constraint.
2. Define the decision variables.
3. Express the objective in terms of the decision variables.
4. Express the constraints in terms of the decision variables.
EXAMPLE: IRON WORKS, INC.

PRODUCT MIX PROBLEM

- Iron Works, Inc. manufactures two products made from steel and receives 2000 pounds of steel each month. It takes 2 pounds of steel to make a unit of product 1, and it takes 3 pounds of steel to make a unit of product 2.

- The unit profits for product 1 and product 2 are $100 and $200, respectively.

- Each month the manufacturer has a contract calling for at least 60 units of product 1.

- At most 720 units of product 2 may be produced by the firm's facilities.

- Find the monthly production quantities of product 1 and product 2 to maximize profit from these products.
Mathematical Model

- Decision Variables:

- Objective - Maximize total monthly profit:
✓ Constraints

✓ The total amount of steel used during monthly production must be at most 2000 pounds

✓ The monthly production quantity of product 1 must be greater than or equal to 60

✓ The monthly production quantity of product 2 must be less than or equal to 720

✓ The production levels cannot be negative:
Mathematical Model Summary:
Giapetto’s woodcarving shop manufactures two types of wooden toys: soldiers and trains.

A soldier sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases Giapetto’s variable labor and overhead costs by $14.

A train sells for $21 and uses $9 worth of raw materials. Each train built increases Giapetto’s variable labor and overhead costs by $10.
The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing.

A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor.

A train requires 1 hour of finishing labor and 1 hour of carpentry labor.

Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours.
Ø Demand for trains is unlimited but at most 40 soldiers are bought each week.
Ø Giapetto wants to maximize weekly profit.
Ø Formulate a mathematical model of Giapetto’s situation.
EXAMPLE: WOOD CARVING SHOP

- Decision Variables:
- Objective function:
- Constraints:
- Model:
- Parameters:
Solution

- **Feasible solution:** The values of the decision variables that satisfy all of the constraints.
- **Optimal solution:** A feasible solution that provides the best objective function value.

Deterministic vs. stochastic models

- **deterministic:** All input data are known exactly and do not vary
- **stochastic:** Any of the uncontrollable inputs is subject to variation
- Generally, stochastic models are more difficult to solve
Three Plants:

- Plant 1: aluminum frame and hardware made
- Plant 2: wood frames made
- Plant 3: produces the glass and assembles the product.

Two new products:

- Product 1: 8-foot glass door with aluminum frame
- Product 2: 4x6 foot double-hung wood framed window

- Product 1 requires processing in Plant 1 and 3.
- Product 2 requires processing in Plant 2 and 3.
- No limits on the sales.
Data:

Each product is produced in batches of 20 units.

Available production time per week

- Plant 1: 4 hours / week
- Plant 2: 12 hours / week
- Plant 3: 18 hours / week
Production time for each batch of products in plants

Product 1: Plant 1: 1 hour / batch
Plant 2: 0 hour / batch
Plant 3: 3 hour / batch

Product 2: Plant 1: 0 hour / batch
Plant 2: 2 hour / batch
Plant 3: 2 hour / batch

Profit per batch of product (dollars per batch)
Product 1: 3000, Product 2: 5000
Decision Variables:

Objective function:

Constraints:
**Standard Form:**

max \[ z = \sum_{j=1}^{n} c_j x_j \]  

s.t. \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i = 1, \ldots, m \]  
[ \quad x_j \geq 0 \quad \forall j = 1, \ldots, n \]

\[ z \text{ : value of overall performance measure} \]
\[ x_j \text{ : level of activity } j \ (j=1,\ldots,n) \]
\[ c_j \text{ : performance measure coefficient for activity } j \]
\[ b_j \text{ : amount of resource } i \text{ available } \ (i=1,\ldots,m) \]
\[ a_{ij} \text{ : amount of resource } i \text{ consumed by each unit of activity } j \]

**Decision Variables:** \( x_j \)

**Parameters:** \( c_j, a_{ij}, b_j \)
An LP model can have the following variations:

1. **Objective Function**: minimization or maximization problem.

2. **Direction of constraints**

   \[ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \leq b_i \quad \forall i = 1, \ldots, m \]  
   less than or equal to

   \[ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \geq b_i \quad \forall i = 1, \ldots, m \]  
   greater than or equal to

   \[ a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n = b_i \quad \forall i = 1, \ldots, m \]  
   equality

3. **Non-negativity constraints**

   \[-\infty \leq x_j \leq \infty\]
Solution: any specification of values for the decision variables, $x_j$, is called a solution

**Infeasible Solution:** a solution for which at least one constraint is violated.

**Feasible Solution:** a solution for which all of the constraints are satisfied

**Corner - Point Feasible (CPF) solution:** a solution that lies at the corner of the feasible region.

**Optimal Solution:** a feasible solution that has the most favorable value of the objective function.
- maximization $\rightarrow$ largest $z$
- minimization $\rightarrow$ smallest $z$

**Multiple Optimal Solutions:** infinite number of solutions with the most favorable value of the objective function

The best CPF = optimal solution
ASSUMPTIONS OF LINEAR PROGRAMMING

1. Proportionality:
   - contribution of each activity to the objective function, $z$, is proportional to its level.
     $$c_jx_j$$
   - contribution of each activity to each functional constraint is proportional to its level.
     $$a_ix_j$$

2. Additivity:
   - every function is the sum of the individual contribution of the respective activities.
     $$z = \sum_{j=1}^{n} c_jx_j \quad \sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad \forall i = 1,\ldots,m$$
ASSUMPTIONS OF LINEAR PROGRAMMING
3. **Divisibility:**
   - decision variables are allowed to have any real values that satisfy the functional and non-negativity constraints.

4. **Certainty:**
   - the parameter values are assumed to be known constants.
SOLVING LP PROBLEMS

- Find the solution that gives the best overall value of the objective function.
\[\begin{align*}
\text{max} \quad & z = 3x_1 + 5x_2 \\
\text{s.t.} \quad & x_1 \leq 4 \\
\quad & 2x_2 \leq 12 \\
\quad & 3x_1 + 2x_2 \leq 18 \\
\quad & x_1, \ x_2 \geq 0
\end{align*}\]
Definition: Adjacent CPF solutions

For any linear programming problem with $n$ decision variables, two CPF solutions are adjacent to each other if they share $n-1$ constraint boundaries. The two adjacent CPF solutions are connected by a line segment that lies on these same shared constraint boundaries. Such a line is referred to as an edge on the feasible region.
<table>
<thead>
<tr>
<th>Point</th>
<th>Coordinates ((x_1, x_2))</th>
<th>Adjacent CPF soln</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(0,0)</td>
<td>B (0,6)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>E (4,0)</td>
</tr>
<tr>
<td>B</td>
<td>(0,6)</td>
<td>A (0,0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C (2,6)</td>
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<tr>
<td>C</td>
<td>(2,6)</td>
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<td>D (4,3)</td>
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</table>
If a CPF solution has no adjacent CPF solutions that are better, then it must be an optimal solution.
OPTIMIZATION ALGORITHM

1. **Initialize**: choose an initial CPF solution.

2. **Optimality Test**: evaluate the performance measure at the current solution. If its value is larger than all of its adjacent CPF solutions; the current solution is optimal. Otherwise go to step 3.

3. **Iteration**: Move to a better adjacent CPF solution. Go to step 2.
**CONSIDERATIONS**

- Simplex method focuses solely on CPF solutions.
- Simplex method is an iteration algorithm.
- Whenever possible, the initialization of the simplex method chooses the origin as the initial CPF solution.
- Given a CPF solution, it is much quicker to gather information about its adjacent CPF solutions than its non-adjacent CPF solutions.
After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution. The most promising edge is selected and this edge is chosen for the next iteration.

A positive rate of improvement in $Z$ implies that the adjacent CPF solution is better; a negative rate of improvement in $Z$ implies that the adjacent CPF solution is worse. The optimality test consists of simply of checking whether any of the edges give a positive rate improvement in $Z$. If none do, then the current CPF solution is optimal.