INDR 262 Introduction to Optimization Methods

STARTING SOLUTION AND CONVERGENCE

**Simplex Method:** Simple method starts from an initial basic feasible solution.

\[
\begin{align*}
\min \ z &= cx \\
st. \quad Ax &\leq b \\
& \quad x \geq 0
\end{align*}
\]

If \( b \geq 0 \)
Add slacks \( x_s \) to each constraint:
\[
Ax + x_s = b \\
x, \ x_s \geq 0
\]
Therefore, an initial basic feasible solution can be easily obtained when we have only \( Ax \leq b \) in the set of constraints.

**Question:** What happens if \( b \) is not nonnegative \( Ax \geq b \) and \( x \geq 0 \), \( b \) must be nonnegative for a feasible solution.
Add artificial variables, \( x_a \) to negative rows (or rows without a designated basic variable).

How to eliminate artificial variables?
Artificial variables should be 0.
Two methods for solving problems without an initial basic feasible solution:
1. The big-M method
2. Two – Phase method

1. **The big-M Method**

\[
\begin{align*}
\max \ c x \\
st. \quad Ax &= b \\
& \quad x \geq 0
\end{align*}
\]

If an initial basic feasible solution is unknown, introduce the artificial variable \( x_a \):
\[
Ax + x_a = b \\
x, \ x_a \geq 0
\]
In order to eliminate artificial variables from the basis, introduce sufficiently large \( M \) (big-M) as cost coefficient of the artificial variables in the objective

\[
\begin{align*}
\max \ z &= cx - Mx_a \\
st. \quad Ax + x_a &= b \\
& \quad x, \ x_a \geq 0
\end{align*}
\]

2. **The Two-Phase Method**

**Phase I:** Formulate the constraint of the LP problem as in the big-M method to secure an initial basic feasible solution. Next, find a basic solution of the resulting equation that minimizes the sum of the artificial variables.
\[ \text{min } z = x_2 \]
\[ \text{s.t. } A^T x + x_2 = b \]
\[ x, x_2 \geq 0 \]

If the optimal solution is positive, the LP problem has no feasible solution. Otherwise, go to phase II.

**Phase II:** Use the feasible solution obtained in Phase I as the initial basic feasible solution. Solve the LP problem with simplex method.

**Example:**

\[ \text{min } z = 3x_1 + 2x_2 + 4x_3 \]

subject to

\[ 2x_1 + x_2 + 3x_3 = 60 \]
\[ 3x_1 + 3x_2 + 5x_3 \geq 120 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]
\[ x_3 \geq 0 \]

Convert the minimization objective to maximization:

\[ \text{max } -z = -3x_1 - 2x_2 - 4x_3 \]

(a) **Solution by the Big M method.**

Rewrite the problem introducing slack \((x_4)\) and artificial variables \((x_5, x_6)\):

\[ \text{max } -z = -3x_1 + 2x_2 + 4x_3 + Mx_5 + Mx_6 = 0 \]
\[ 2x_1 + x_2 + 3x_3 + \frac{x_5}{3} = 60 \]
\[ 3x_1 + 3x_2 + 5x_3 - x_4 + x_6 = 120 \]

Since the artificial variables \(x_5\) and \(x_6\) are basic, we must reorganize row 0 to have 0 reduced costs for these variables:

New row 0 = Old row 0 – \(M\)\(\{\text{row 1}\}\) – \(M\)\(\{\text{row 2}\}\)

\[ = \begin{bmatrix} 3 & 2 & 4 & 0 & M & M & \mid & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 & 0 & 1 & 0 & \mid & 60 \end{bmatrix} - \begin{bmatrix} 3 & 3 & 5 & -1 & 0 & 1 & \mid & 120 \end{bmatrix} \]

\[ = \begin{bmatrix} -5M+3 & -4M+2 & -8M+4 & M & 0 & 0 \end{bmatrix} \]

<table>
<thead>
<tr>
<th>Iter</th>
<th>B.V.</th>
<th>Eqn</th>
<th>(z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>RHS</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
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<td>(Z)</td>
<td>0</td>
<td>-1</td>
<td>-5M+3</td>
<td>-4M+2</td>
<td>-8M+4</td>
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<td>0</td>
<td>0</td>
<td>-180M</td>
<td></td>
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<td>1</td>
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<td>20</td>
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<td>3</td>
<td>3</td>
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<td>(\frac{3}{5}M+\frac{1}{5})</td>
<td>(-\frac{1}{5}M+\frac{3}{5})</td>
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<td>(\frac{1}{5}M+\frac{4}{5})</td>
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<td>(-\frac{1}{3})</td>
<td>4/3</td>
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<td>15</td>
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<tr>
<td>2</td>
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<td>0</td>
<td>1</td>
<td>(\frac{1}{4})</td>
<td>(\frac{3}{4})</td>
<td>-(\frac{1}{4})</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_2)</td>
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<td>0</td>
<td>(-\frac{1}{4})</td>
<td>1</td>
<td>0</td>
<td>(-\frac{1}{4})</td>
<td>-5/4</td>
<td>(\frac{3}{4})</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
The optimal solution: \( x_1^* = 0, x_2^* = 15, x_3^* = 15, z^* = 90 \)

b) Solution by the Two-Phase method.

**Phase I:**

\[ \begin{align*}
\text{min} & \quad z = x_5 + x_6 \\
\text{subject to} & \quad \\
2x_1 + x_2 + 3x_3 + x_5 & = 60 \\
3x_1 + 3x_2 + 5x_3 - x_4 + x_6 & = 120 \\
x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0
\end{align*} \]

Convert the minimization objective to maximization:

\[ \begin{align*}
\text{min} & \quad z = x_5 + x_6 \\
\text{max} & \quad -z = -x_5 - x_6
\end{align*} \]

(0) \[ -z + x_5 + x_6 = 0 \]

(1) \[ 2x_1 + x_2 + 3x_3 + x_5 = 60 \]

(2) \[ 3x_1 + 3x_2 + 5x_3 - x_4 + x_6 = 120 \]

Phase II: Rewrite the optimization model by,

- dropping the artificial variables \( x_5 \) and \( x_6 \)
- using the original objective function

\[ \begin{align*}
\text{min} & \quad z = 3x_1 + 2x_2 + 4x_3 \\
\text{s.t.} & \quad \\
\frac{3}{4}x_1 + x_3 + \frac{1}{4}x_4 & = 15 \\
-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 & = 15 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*} \]
Convert the minimization objective to maximization:

\[
\begin{align*}
\text{min} \quad z &= 3x_1 + 2x_2 + 4x_3 \\
\text{max} \quad -z &= -3x_1 - 2x_2 - 4x_3
\end{align*}
\]

s.t.

\[
\begin{align*}
\frac{3}{4}x_1 + x_3 + \frac{1}{4}x_4 &= 15 \\
-\frac{1}{4}x_1 + x_2 - \frac{3}{4}x_4 &= 15 \\
x_1, \ x_2, \ x_3, \ x_4 &\geq 0
\end{align*}
\]

\[
\begin{align*}
(0) & \quad -z + 3x_1 + 2x_2 + 4x_3 = 0 \\
(1) & \quad \frac{3}{4}x_1 + x_3 + \frac{1}{4}x_4 = 15 \\
(2) & \quad -\frac{1}{4}x_1 + x_2 - \frac{3}{4}x_4 = 15
\end{align*}
\]

Since the variables \(x_2\) and \(x_3\) are basic, we must reorganize row 0 to have 0 reduced costs for these variables:

New row 0

\[
\begin{align*}
\text{Old row 0} & \quad -z + 3x_1 + 2x_2 + 4x_3 = 0 \\
= [3 \ 2 \ 4 \ 0 \ | \ 0] - 4*[\frac{3}{4} \ 0 \ 1 \ \frac{1}{4} \ | \ 15] - 2*[-\frac{1}{4} \ 1 \ 0 \ -\frac{3}{4} \ | \ 15] \\
= [\frac{1}{2} \ 0 \ 0 \ \frac{1}{2} \ | \ -90]
\end{align*}
\]

<table>
<thead>
<tr>
<th>Iter</th>
<th>B.V.</th>
<th>Eqn</th>
<th>(z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td>-1</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>-90</td>
</tr>
<tr>
<td></td>
<td>(x_3)</td>
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<td>0</td>
<td>1/4</td>
<td>0</td>
<td>1</td>
<td>1/4</td>
<td>15</td>
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<td></td>
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<td>0</td>
<td>-1/4</td>
<td>1</td>
<td>0</td>
<td>-1/4</td>
<td>15</td>
</tr>
</tbody>
</table>

The optimal solution: \(x_1^* = 0, \ x_2^* = 15, \ x_3^* = 15, \ z^* = 90\)
Analysis of the big-M Method

\[
\text{max } z = cx - Mx_d \\
\text{s.t. } Ax + x_d = b \\
x, \quad x_d \geq 0
\]

- Optimal solution is found
- The problem is unbounded. \( z_k - c_k = \min (z_j - c_j) > 0, y_k \leq 0 \)

\[ x^* \geq 0, x_d^* = 0 \]
Artificial values have all 0 values. The solution is optimal.

\[ x^* \geq 0, x_d^* \neq 0 \]
Some of the artificial variables are nonzero. The solution is infeasible.

\[ x^* \geq 0, x_d^* = 0 \]
Unbounded

\[ x^* \geq 0, x_d^* \neq 0 \]
Infeasible
Degeneracy

When there is a tie in the minimum ratio test for determining the leaving variable, one or more of the basic variables will be zero in the next iteration, the new solution is called a degenerate solution.

→ The model has at least one redundant constraint.

Ex: max z = 3x₁ + 9x₂
    s.t.  x₁ + 4x₂ ≤ 8
          x₁ + 2x₂ ≤ 4
          x₁, x₂ ≥ 0

Initial Simplex Tableau:

<table>
<thead>
<tr>
<th>x₄</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>RHS</th>
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<tbody>
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<tr>
<td>x₂</td>
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<tr>
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<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

x₃ leaves x₂ enters

<table>
<thead>
<tr>
<th>x₄</th>
<th>z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>0</td>
<td>-3/4</td>
<td>0</td>
<td>9/4</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>x₂</td>
<td>0</td>
<td>1/4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>x₄</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>-1/2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

x₄ leaves x₁ enters

<table>
<thead>
<tr>
<th>x₄</th>
<th>z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>3/2</td>
<td>18</td>
</tr>
<tr>
<td>x₂</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>2</td>
</tr>
<tr>
<td>x₁</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Implications of degenerate solutions:
   a) cycling
   b) circling
Unbounded Solutions:

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraint. This condition is called unsoundness.

Ex: \[ \text{max } z = x_1 + 2x_2 \]
\[ \text{s.t. } \]
\[ x_1 - x_2 \leq 10 \]
\[ 2x_1 \leq 40 \]
\[ x_1, x_2 \geq 0 \]

<table>
<thead>
<tr>
<th>( x_B )</th>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
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<td>-1</td>
<td>1</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>( x_4 )</td>
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<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>40</td>
</tr>
</tbody>
</table>

Observations:
- \( x_2 \) is selected as the entering basic variable since it has the most negative row-0 coefficient
- \( x_3 \) cannot leave the basis because \( a_{12} = -1 \) indicating that the value of the entering basic variable, \( x_2 \), should decrease which is not possible since the current value of \( x_2 \) which is a nonnegative variable is 0; decreasing \( x_2 \) further will make the solution infeasible
- \( x_4 \) cannot leave the basis because \( a_{12} = 0 \) indicating that the value of the entering basic variable, \( x_2 \), can increase to infinity without violating this constraint

Therefore, this problem is unbounded.
Example 1:

\[
\begin{align*}
\text{min } & \quad z = 4x_1 + x_2 \\
\text{s.t. } & \quad 3x_1 + x_2 = 3 \\
& \quad 4x_1 + 3x_2 \geq 6 \\
& \quad x_1 + 2x_2 \leq 4 \\
& \quad x_1, \quad x_2 \geq 0
\end{align*}
\]

First introduce the slack variables:

\[
\begin{align*}
\text{min } & \quad z = 4x_1 + x_2 \\
\text{s.t. } & \quad 3x_1 + x_2 = 3 \\
& \quad 4x_1 + 3x_2 - x_3 = 6 \\
& \quad x_1 + 2x_2 + x_4 = 4 \\
& \quad x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0
\end{align*}
\]

No initial basis, introduce artificial variables.

a) big-M Method:

\[
\begin{align*}
\text{min } & \quad z = 4x_1 + x_2 + Mx_5 + Mx_6 \\
\text{s.t. } & \quad 3x_1 + x_2 + \frac{x_5}{x_6} = 3 \\
& \quad 4x_1 + 3x_2 - x_3 + x_6 = 6 \\
& \quad x_1 + 2x_2 + x_4 = 4 \\
& \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6 \geq 0
\end{align*}
\]

Eliminate inconsistency in basic variable \(x_5\) and \(x_6\).
Multiply each row whose basic variable is an artificial variable by the negative of its row-0 entry and add to row-0:

**New row 0:** **Old row 0** +M*row 1 +M*row 2

<table>
<thead>
<tr>
<th>(x_B)</th>
<th>Eqn</th>
<th>(z)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>RHS</th>
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<tbody>
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<td>-1</td>
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<td>-M</td>
<td>-M</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>(x_6)</td>
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<td>3</td>
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<td>0</td>
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<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>4</td>
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</tbody>
</table>
Entering Variable: $x_1$ has the most positive coefficient in row-0.
Leaving Variable: minimum ratio test $\rightarrow x_2$ is the leaving basic variable

<table>
<thead>
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<th>$x_B$</th>
<th>Eqn</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>RHS</th>
<th>Ratio</th>
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<td>0</td>
<td>0</td>
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<td>-1/3(7M+4)</td>
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<td>0</td>
<td>1/3</td>
<td>0</td>
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<td>3</td>
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<td>$x_6$</td>
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<td>0</td>
<td>5/3</td>
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<td>6/5</td>
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<td>$x_4$</td>
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<td>1/3</td>
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<td>9/5</td>
</tr>
</tbody>
</table>

Entering Variable: $x_2$ has the most positive coefficient in row 0.
Leaving Variable: minimum ratio test $\rightarrow x_6$ is the leaving basic variable

<table>
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<th>Eqn</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
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<td>0</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>-3/5</td>
<td>0</td>
<td>-4/5</td>
<td>3/5</td>
<td>3/5</td>
</tr>
<tr>
<td>$x_4$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

Continue until finding the optimal solution!

b) Two-Phase Method:
Phase I: min $z_0 = \frac{x_5 + x_6}{3x_1 + x_2}$
\[ 3x_1 + x_2 = 3 \]
\[ 4x_1 + 3x_2 - x_3 + x_4 = 6 \]
\[ x_1 + 2x_2 = 4 \]
\[ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \]

<table>
<thead>
<tr>
<th>$x_B$</th>
<th>$z_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_0$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Old $z_0$-row : 0 0 0 -1 -1 0 | 0
+1*$x_5$-row : 3 1 0 1 0 0 | 3
+1*$x_6$-row : 4 3 -1 0 1 0 | 6

New $z_0$ - row : 7 4 -1 0 0 0 | 9

<table>
<thead>
<tr>
<th>$x_B$</th>
<th>$z_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_0$</td>
<td>1</td>
<td>7</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>
→ Simplex Method: Optimal Solution for the Phase I problem

\[
\begin{array}{cccccccc}
  & x_B & z_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \text{RHS} \\
 z_0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
x_1 & 0 & 1 & 0 & 1/5 & 0 & 3/5 & -1/5 & 3/5 \\
x_2 & 0 & 0 & 1 & -3/5 & 0 & -4/5 & 3/5 & 6/5 \\
x_4 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 \\
\end{array}
\]

→ Optimal Solution with \( z_0 = 0 \)
→ Continue with Phase II.

**Phase II:** Delete artificial variables and use the optimal solution in Phase I as the LP problem:

\[
\begin{align*}
  \text{min} \quad & z = 4x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 1/5x_3 = 3/5 \\
& x_2 - 3/5x_3 = 6/5 \\
& x_3 + x_4 = 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

\[
\begin{array}{cccccc}
  & x_B & z & x_1 & x_2 & x_3 & \text{RHS} \\
 z & 1 & -4 & -1 & 0 & 0 & 0 \\
x_1 & 0 & 1 & 0 & 1/5 & 0 & 3/5 \\
x_2 & 0 & 0 & 1 & -3/5 & 0 & 6/5 \\
x_4 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

\( x_1 \) and \( x_2 \) have nonzero coefficients in the row-0, they must be substituted out.

Old row-0 : -4 -1 0 0 | 0
+4\( *x_1 \)-row : 4 0 1/5 0 | 12/5
+1\( *x_2 \)-row : 0 1 -3/5 0 | 6/5

New row – 0: 0 0 1/5 0 | 18/5

→ Initial simplex tableau:

\[
\begin{array}{cccccc}
  & x_B & z & x_1 & x_2 & x_3 & \text{RHS} \\
 z & 1 & 0 & 0 & 1/5 & 0 & 18/5 \\
x_1 & 0 & 1 & 0 & 1/5 & 0 & 3/5 \\
x_2 & 0 & 0 & 1 & -3/5 & 0 & 6/5 \\
x_4 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

Continue until finding the optimal solution!