DUALITY

For every LP problem, there is a dual problem. The dual problem may be used to obtain a solution to the original problem (primal) and the variables used in the dual problem can give useful information about the optimal solution to the LP.

\[ \text{Primal LP:} \quad \begin{align*}
\text{Min:} & \quad \min Z = cx \\
& \text{s.t.} \quad Ax \geq b \\
& \quad x \geq 0
\end{align*} \]

\[ \text{Dual LP:} \quad \begin{align*}
\text{Max:} & \quad \max W = y^T b \\
& \text{s.t.} \quad y^T A \leq c \\
& \quad y \geq 0
\end{align*} \]

**Primal and Dual Relationships**

<table>
<thead>
<tr>
<th>Min Problem</th>
<th>Max Problem</th>
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<tbody>
<tr>
<td>Constraints</td>
<td>Variables</td>
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<tr>
<td>≥</td>
<td>≥ 0</td>
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<td>≤</td>
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<tr>
<td>=</td>
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<tr>
<td>Variables</td>
<td>Constraints</td>
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<td>≥</td>
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<tr>
<td>unrestricted</td>
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**Example 1:**

min \( 2x_1 + 3x_2 + 4x_3 \)

s.t. \( x_1 + 5x_2 + 6x_3 \geq 7 \) / \( y_1 \)

\( 8x_1 + 9x_2 + 10x_3 \geq 11 \) / \( y_2 \)

\( x_1, x_2, x_3 \geq 0 \)

Dual: max \( 7y_1 + 11y_2 \)

s.t. \( y_1 + 8y_2 \leq 2 \)

\( 5y_1 + 9y_2 \leq 3 \)

\( 6y_1 + 10y_2 \leq 4 \)

\( y_1, y_2 \geq 0 \)

**Example 2:**

max \( 5x_1 + 12x_2 + 4x_3 \)

s.t. \( x_1 + 2x_2 + x_3 \leq 10 \) / \( y_1 \)

\( 2x_1 - x_2 + 3x_3 = 8 \) / \( y_2 \)

\( x_1, x_2, x_3 \geq 0 \)
Dual: min $10y_1 + 8y_2$
    s.t. $y_1 + 2y_2 \geq 5$
        $2y_1 - y_2 \geq 12$
        $y_1 + 3y_2 \geq 4$
        $y_1 \geq 0$, $y_2 = \text{unrestricted}$

**Weak duality:** The objective function value for any feasible solution to the minimization problem is always greater than or equal to the objective function value for any feasible solution to the maximization problem.

**Optimality:** If $x_0$ and $y_0$ are feasible solutions to the primal and dual problems such that $cx_0 = y_0 b$ then $x_0$ and $y_0$ are optimal solutions to their respective problems.

**KKT Optimality Conditions**

Primal: min $cx$
    s.t. $Ax \geq b$
        $x \geq 0$

Dual: max $yb^T$
    s.t. $yA^T \leq c^T$
        $y \geq 0$

1. $Ax^* \geq b$, $x^* \geq 0$  Primal feasibility
2. $y^*A^T \leq c^T$, $y^* \geq 0$  Dual feasibility
3. $y^*(Ax^*-b) = 0$
   $y^*b^T = cx^*$

**Fundamental Theorem of Duality**

Exactly one of the following statements is true for primal-dual relationship
1. Both have optimal solution $x^*$ and $y^*$ with $cx^* = y^*b^T$
2. One problem has unbounded objective value, the other must be infeasible
3. Both problems are infeasible

**Complementary Slackness**

If a variable in one problem is positive (i.e., basic variable), then the corresponding constraint in the other problem must be tight. Conversely, if a constraint in one problem is not tight, then the corresponding variable in the other problem must be zero.
Economic Interpretation of Duality

The LP problem can be considered as a resource allocation model with an objective of minimizing the cost subject to availability of limited resources.

Primal: \( \text{min } cx \)  
\[ \text{s.t. } Ax \geq b \]  
\[ x \geq 0 \]

Dual: \( \text{max } y^T b \)  
\[ \text{s.t. } y^T A \leq c^T \]  
\[ y \geq 0 \]

The problem has \( m \) resources and \( n \) activities.

The coefficient \( c_j \) in the primal represents the cost per unit of activity \( j \). Resource \( i \), whose maximum availability is \( b_i \), is consumed at the rate of \( a_{ij} \) per unit of activity \( j \).

Interpretation of Dual Variables

We know from weak duality that the objective function values are related; and from KKT that at the optimal solution, the objective values are equal to each other:

\[ cx^* = y^* b^T \]

Given that the primal problem represents a resource allocation model, we can think of \( \sum_j c_j x_j \) as representing cost. Because \( b_i \) represents the number of units available of resource \( i \), the dual variables \( y_i \) must represent the worth per unit of resource \( i \). In general \( y_i \) are referred as “dual prices” - “shadow prices”

Weak duality:

<table>
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<tr>
<th>Primal</th>
<th>Dual</th>
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<tbody>
<tr>
<td>Cost</td>
<td>Worth of Resources</td>
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As long as the cost is greater than the worth of resources, the corresponding primal and dual solutions cannot be optimal. Optimality is attained only when all of the resources have been exploited completely.
Interpretation of the Dual Constraints

Property: at any primal iteration

\[
\begin{align*}
\text{Objective function coefficient of variable } j \text{ in one problem} & = \text{Left-hand side minus right-hand side of constraint } j \text{ in the other problem} \\
\end{align*}
\]

(objective coefficient of \( x_j \)) = \( \sum_{i=1}^{m} a_{ij} y_i - c_j \)

\( y_i \) represents the imputed cost per unit of resource \( i \), and the quantity \( \sum_{i=1}^{m} a_{ij} y_i \) is the imputed cost of all resources needed to produce 1 unit of activity \( j \).

\[
\Rightarrow \left\{ \begin{array}{l}
\text{Cost per unit of activity } j \\
\text{Imputed cost of used resources}
\end{array} \right. \left\{ \begin{array}{l}
\text{per unit of activity } j
\end{array} \right.
\]

The minimization optimality condition of the simplex method says that an increase in the level of an unused activity \( j \) can improve cost only if its objective coefficient is positive.

Define \( z_j = \sum_{i=1}^{m} a_{ij} y_i \) to represent imputed cost of used resources per unit of activity \( j \).

Remember \( z_j - c_j \) in the simplex tableau. \( z_j - c_j \) is the reduced cost of activity \( j \).