1. Let $G$ be an abelian group and $T = \{x \in G : \exists n \in \mathbb{N} \text{ such that } x^n = e\}$. Prove that $T$ is a subgroup of $G$ ($T$ is known as the torsion subgroup of $G$).

2. Prove that if $H$ and $K$ are subgroups of an abelian group $G$, then $\{hk | h \in H \text{ and } k \in K\}$ is a subgroup of $G$.

3. Let $G$ be a group, $a \in G$, and define $H_a = \{x \in G : xa = ax\}$. Prove that $H_a$ is a subgroup of $G$. ($H_a$ is known as the centralizer of $a$. The notation $C(a)$ is also used for $H_a$)

4. Let $S$ be any subset of a group $G$.
   a) Prove that $H_S = \{x \in G | xs = sx \text{ for all } s \in S\}$ is a subgroup of $G$.
   b) Prove that $H_G$ is an abelian group. ($H_G$ is known as the center of $G$.)

5. Prove that every cyclic group is abelian.

6. Prove that a group with no proper nontrivial subgroup is cyclic.

7. Prove that a cyclic group with only one generator can have at most 2 elements.

8. Prove that if $H$ and $K$ are subgroups of the group $G$, then $H \cap K$ is a subgroup of $G$.

9. Give an example of a group and subgroups of $H, K, L$ of $G$ such that $H \cup K \cup L$ is a subgroup of $G$.

10. Let $G$ be a group and $A, B, C$ be subgroups of $G$ with $A$ is a subgroup of $C$. Prove that $A(B \cup C) = AB \cup C$.

11. Let $G$ be a group and $H$ a subgroup of $G$ satisfying: whenever $Ha \neq Hb$ then $aH \neq bH$, for $a, b \in G$. Prove that $gHg^{-1} \subseteq H$ for all $g \in G$.

12. Find the right cosets and the left cosets of the subgroup $(H_n, +)$ of $(\mathbb{Z}, +)$ where $H_n$ is the set of all multiples of $n$. 