MATH 206 ALGEBRA II (SPRING 2003)
HOMEWORK ASSIGNMENT I

1. Prove that \( \mathbb{Z}[\sqrt{2}] = \{ n + \sqrt{2}m : n, m \in \mathbb{Z} \} \), for the usual addition and product of real numbers, is a ring but not a field.

2. Prove that \( \mathbb{Q}[\sqrt{2}] = \{ n + \sqrt{2}m : n, m \in \mathbb{Q} \} \), for the usual addition and product of real numbers, is a field.

3. Prove that \( \mathbb{Z}[\sqrt{2}] \) is the smallest subring of \( \mathbb{R} \) containing \( \mathbb{Z} \) and \( \sqrt{2} \). (A subring of a ring is a subset of the ring that is a ring under induced operations from the whole ring.)

4. Consider the matrix ring \( M_2(\mathbb{Z}_2) \). Find the order of the ring, that is, the number of elements in it. List all units, in the ring.

5. Give an example of a ring with unity \( 1 \neq 0 \) that has a subring with nonzero unity \( 1' \neq 1 \).

6. Prove that if \( U \) is the collection of all units in a ring \((R, +, \cdot)\) with unity, then \((U, \cdot)\) is a group.

7. An element \( a \) of a ring \( R \) is nilpotent if \( a^n = 0 \) for some \( n \in \mathbb{Z}^+ \). Prove that if \( a \) and \( b \) are nilpotent elements of a commutative ring, then \( a + b \) is also nilpotent.

8. Prove that a subset \( S \) of a ring \( R \) gives a subring of \( R \) if and only if the following hold:
   - \( 0 \in S \);
   - \( (a - b) \in S \) for all \( a, b \in S \);
   - \( ab \in S \) for all \( a, b \in S \).

9. Let \( R \) be a ring, and let \( a \) be a fixed element of \( R \). Let \( I_a = \{ x \in R | ax = 0 \} \). Prove that \( I_a \) is a subring of \( R \).

10. a) Prove that an intersection of subrings of a ring \( R \) is again a subring of \( R \).
    b) Prove that an intersection of subfields of a field \( F \) is again a subfield of \( F \).

11. Let \( R \) be a ring that contains at least two elements. Suppose for each nonzero \( a \in R \), there exists a unique \( b \in R \) such that \( aba = a \).
    a) Prove that \( R \) has no zero divisors. (Hint: Start from \( ac = 0 \) or \( ca = 0 \), where \( a \neq 0 \) and then consider \( a(b + c)a \))
    b) Prove that \( bab = b \).
    c) Prove that \( R \) has unity.
    d) Prove that \( R \) is a division ring.