Math 302. Homework 4

Solve the following problems:

(1) True or false? Why?
   a) A sequence of decreasing closed balls $B_1 \supset B_2 \supset \cdots$ in a complete metric space has a nonempty intersection.
   b) A sequence of decreasing open balls $B_1 \supset B_2 \supset \cdots$ in a complete metric space has a nonempty intersection.

(2) Show that the set of functions $u(x) = kx^4$, $k \in [0, 4]$ is compact in $C[0, 1]$.

(3) Give an example of a sequence of nonempty closed and bounded sets $\{A_n\}$ of a complete metric space $X$ such that
   a) $A_{n+1} \subset A_n$, $n = 1, 2, \cdots$
   b) $\cap_{n=1}^{\infty} A_n = \emptyset$.

(4) Let $\lambda \in \mathbb{R}$ is a given parameter and $K \in \tilde{L}^2[\Pi]$, $\Pi = [a, b] \times [a, b]$ is a given function.
   - Show that the operator
     $$Tu(x) = \lambda \int_a^b K(x, y)u(y)dy$$
     maps $\tilde{L}^2[a, b]$ to $\tilde{L}^2[a, b]$.
   - find a number $a_0 > 0$ for which the operator $T$ is a contraction when $|\lambda| < a_0$.
   - has the operator $T$ a fixed point in $\tilde{L}^2[a, b]$ for $|\lambda| < a_0$?

(5) Prove that the space $\ell^p$, $p \geq 1$ is a Banach space.