Math 302. Homework 7b

Solve the following problems:

1. True or false? Why?
   (1) The function
   \[
   f(x) = \begin{cases} 
   x^3, & x \text{ - irrational,} \\
   1, & x \text{ - rational}
   \end{cases}
   \]
   is Lebesgue integrable on \([0, 1]\).

2. If a function \(f\) is bounded on \([0, 1]\) and Lebesgue integrable, then the functions
   \([f(x)]^4, |f(x)|, \frac{1}{f(x)}\) are Lebesgue integrable on \([0, 1]\).

3. If \(f\) is continuous on \([0, 1]\) and
   \[
   \int_0^1 x^k f(x) dx = 0, \quad k = 1, 2, \ldots
   \]
   then \(f \equiv 0\) on \([0, 1]\).

4. If \(f\) is Lebesgue integrable on \([0, 1]\) and
   \[
   \int_0^1 x^k f(x) dx = 0, \quad k = 1, 2, \ldots
   \]
   then \(f = 0\) a.e. in \([0, 1]\).

5. Let \(\{f_n\} \subset L^p[a, b], 1 < p < \infty, \{f_n\}\) converges in \(L^p[a, b]\) to \(f\), and let \(\{g_n\}\) is a sequence of measurable functions such that \(|g_n| \leq M, n = 1, 2, \ldots\) and \(g_n \to g\) a.e. Then
   \[f_n g_n \to \text{ in } L^p[a, b].\]