9. Relations

Relations are discrete structures that are used to represent relationships between elements of sets.

Relations can be used to solve problems such as:
- Determining which pairs of cities are linked by airline flights in a network,
- Computing the distance between a pair of registered Facebook users,
- Finding an efficient order for different phases of a complicated project,
- Producing a useful way to store information in computer databases, etc.
9.1 Relations and Their Properties

**Definition: Binary relation**
Let $A$, $B$ be sets. A binary relation $R$ from $A$ to $B$ is a set of ordered pairs, hence a subset of $A \times B$.

**Notation:**
- $a$ is “related to” $b$ by $R$: $a \ R \ b : \ (a,b) \in R; \ a \in A, \ b \in B$
- $a$ is “not related to” $b$ by $R$: $a \not\mathrel{R} b : \ (a,b) \notin R$
e.g.

$A$: set of cities

$B$: set of countries

$R$: $(a, b) \in R$ if city $a$ is in country $b$.

$(\text{Izmir, Turkey}), (\text{Paris, France}) \in R$
**Function is a special case of relation**

A function $f$ from $A$ to $B$ can be thought of as the set of ordered pairs $(a, b)$ s.t. $b = f(a)$

Since the function $f$ is a subset of $A \times B$, $f$ is a relation from $A$ to $B$.

Function is a special case of relation: Every element of $A$ is the first element of exactly one ordered pair of the function $f$. 

\[
\begin{array}{c}
0 \xrightarrow{f} a \\
1 \xrightarrow{f} b \\
2 \xrightarrow{f} \end{array}
\]
Relations defined on a single set:

Definition:
A relation on a set \( A \) is a relation from \( A \) to \( A \).

e.g.
\[
A = \{1, 2, 3, 4\} \\
R = \{(a, b) \mid a \mid b, (a, b) \in A \times A\} \\
= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}
\]
Relations defined on a single set:

Definition:
A relation on a set $A$ is a relation from $A$ to $A$.

e.g.
\[
A = \{1, 2, 3, 4\}
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\[
R = \{(a, b) \mid a \mid b, (a, b) \in A \times A\}
\]
\[
= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}
\]

e.g.
How many relations are there on a set with $n$ elements?
\[
|A \times A| = n^2
\]
\[
\therefore 2^n \text{ (# of subsets of } A \times A)\]
Properties of Relations defined on a set:

Definition:
A relation $R$ on a set $A$ is called reflexive iff

$$(a, a) \in R \quad \forall a \in A$$

e.g.

$A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 2), (1, 3)\}$  (not reflexive)

$R_2 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$  (reflexive)

$R_3 = \{(1, 3), (3, 1)\}$  (irreflexive)
**Properties of Relations defined on a set:**

**Definition:**
A relation $R$ on a set $A$ is called **reflexive** iff

$$(a, a) \in R \quad \forall a \in A$$

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$R_1 = \{(1, 2), (2, 2), (1, 3)\}$ \hspace{1em} (not reflexive)

$R_2 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$ \hspace{1em} (reflexive)

$R_3 = \{(1, 3), (3, 1)\}$ \hspace{1em} (irreflexive)

e.g.

$R$: The set of pairs of people having the same eye color \hspace{1em} (reflexive)
Definition:
A relation $R$ on a set $A$ is called

**symmetric** iff the following holds

$$(b,a) \in R \implies (a,b) \in R \quad \forall a, b \in A$$
**Definition:**
A relation $R$ on a set $A$ is called

- **symmetric** iff the following holds
  \[(b,a) \in R \rightarrow (a,b) \in R \quad \forall a, b \in A\]

- **anti-symmetric** iff the following holds
  \[(a,b) \in R \text{ and } (b,a) \in R \rightarrow a = b \quad \forall a, b \in A\]
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**anti-symmetric** iff the following holds

$$(a,b) \in R \text{ and } (b,a) \in R \rightarrow a = b \quad \forall a,b \in A$$

e.g.

$R_t = \{(a,b) \mid a \text{ is taller than } b\}$ anti-symmetric

$R = \{(a,b) \mid a+b+ab = 12; \ a,b \in \mathbb{Z}\}$ symmetric
**Definition:**
A relation $R$ on a set $A$ is called

**symmetric** iff the following holds  
$$(b,a) \in R \rightarrow (a,b) \in R \quad \forall a,b \in A$$

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e.g.
$R_t = \{(a,b) \mid a \text{ is taller than } b\}$  anti-symmetric

$R = \{(a,b) \mid a+b+ab = 12; \, a,b \in \mathbb{Z}\}$  symmetric

**asymmetric** iff $\forall a,b \in A \ (a,b) \in R \rightarrow (b,a) \notin R$
**Definition:**

$R$ on set $A$ is called **transitive** iff

$$(a,b) \in R \text{ and } (b,c) \in R \rightarrow (a,c) \in R \quad \forall a,b,c \in A.$$ 

**e.g.**

$$R_t = \{(a,b) \mid a \text{ is taller than } b\} \text{ transitive?}$$
Definition:
$R$ on set $A$ is called **transitive** iff
$$(a, b) \in R \quad \text{and} \quad (b, c) \in R \quad \rightarrow \quad (a, c) \in R \quad \forall a, b, c \in A.$$ 

*e.g.*
$$R_t = \{(a, b) \mid a \text{ is taller than } b\} \text{ transitive?}$$

*e.g.*
$$A = \{1, 2, 3\}$$

$$R_1 = \{(1, 2), (2, 3), (1, 3)\} \quad \text{(transitive)}$$

$$R_2 = \{(1, 2), (2, 3)\} \quad \text{(not transitive)}$$

$$R_3 = \{(1, 2)\} \quad (?)$$
e.g.
How many reflexive relations are there on a set with \( n \) elements?

If \( R \) is reflexive, then:
- there are \( n \) pairs such that \((a, a) \in R\)
- and \( n(n-1) \) pairs such that \((a, b) \in R\) where \( a \neq b \)

\[ \Rightarrow \text{# of reflexive relations} = 2^{n(n-1)} \]

e.g.
How many symmetric relations are there on a set with \( n \) elements? (Exercise)
**Combining relations:**

Let \( A = \{a, b\} \) \quad B = \{1, 2, 3\}

\[ R_1 = \{(a, 1), (b, 3)\} \]
\[ R_2 = \{(a, 1), (a, 2), (b, 1), (b, 2)\} \]
\[ R_3 = \{(b, 1), (b, 2)\} \]
\[ R_4 = \{(a, 1), (b, 2)\} \]

\[ R_1 \cup R_3 = \{(a, 1), (b, 1), (b, 2), (b, 3)\} \]
\[ R_1 \cap R_2 = \{(a, 1)\} \]
\[ R_2 \setminus R_3 = \{(a, 1), (a, 2)\} \]
\[ R_1 \oplus R_4 = \{(b, 2), (b, 3)\} \]

\( \oplus \) is called “symmetric difference”, acts like XOR
**Definition:** Let $R: A \to B$ and $S: B \to C$. Then the **composite relation** of $R$ and $S$,

$$S \circ R: A \to C$$

is defined s.t.

$$(a, c) \in S \circ R \text{ iff } (a, b) \in R \text{ and } (b, c) \in S.$$
**Definition:** Let $R: A \rightarrow B$ and $S: B \rightarrow C$. Then the **composite relation** of $R$ and $S$,

$S \circ R: A \rightarrow C$ is defined s.t.

$$(a, c) \in S \circ R \text{ iff } (a, b) \in R \text{ and } (b, c) \in S.$$

**Definition:**
Let $R$ be a relation on $A$.
The **powers** $R^n$, $n = 1, 2, 3, \ldots$, are defined by

$R^1 = R, \quad R^2 = R \circ R, \ldots \quad R^n = R^{n-1} \circ R.$

**e.g.**

$R = \{(a,b) \mid b \text{ is a parent of } a\}$

$\Rightarrow R^2 = \{(a,c) \mid c \text{ is a grand-parent of } a\}$ why?

since $(a,b) \in R$ means “$b$ is a parent of $a$”, and $(b,c) \in R$ means “$c$ is a parent of $b$”.
**Theorem:**

$R$ on a set $A$ is transitive iff $R^n \subseteq R$ for all $n = 1, 2, 3, \ldots$
**Theorem:**
$R$ on a set $A$ is transitive iff $R^n \subseteq R$ for all $n = 1, 2, 3,\ldots$

**Proof:**

If part: (if $R^n \subseteq R$ for $n = 1, 2, 3,\ldots$, then $R$ is transitive)
If $R^n \subseteq R$, in particular $R^2 \subseteq R$.
Then, if $(a,b) \in R$ and $(b,c) \in R$, by definition $(a,c) \in R^2$. Since $R^2 \subseteq R$, $(a,c) \in R$. 
∴ $R$ is transitive.

Only if part: (If $R$ is transitive, then $\forall n \ R^n \subseteq R$) Use induction on $n$. 


Theorem:
\( R \) on a set \( A \) is transitive iff \( R^n \subseteq R \) for all \( n = 1, 2, 3, \ldots \)

Proof:

If part: (if \( R^n \subseteq R \) for \( n = 1, 2, 3, \ldots \), then \( R \) is transitive)
If \( R^n \subseteq R \), in particular \( R^2 \subseteq R \).
Then, if \((a,b)\in R \) and \((b,c)\in R \), by definition \((a,c)\in R^2 \). Since \( R^2 \subseteq R \), \((a,c)\in R \).
\( \therefore \) \( R \) is transitive.

Only if part: (If \( R \) is transitive, then \( \forall n \ R^n \subseteq R \)) Use induction on \( n \).
Basis step: \( R^1 \subseteq R \); true for \( n = 1 \).
Inductive step: Assume \( R^n \subseteq R \) and \( R \) is transitive. Show \( R^{n+1} \subseteq R \).
Theorem:
$R$ on a set $A$ is transitive iff $R^n \subseteq R$ for all $n = 1, 2, 3,\ldots$

Proof:

If part: (if $R^n \subseteq R$ for $n = 1, 2, 3,\ldots$, then $R$ is transitive)
If $R^n \subseteq R$, in particular $R^2 \subseteq R$.
Then, if $(a, b) \in R$ and $(b, c) \in R$, by definition $(a, c) \in R^2$. Since $R^2 \subseteq R$, $(a, c) \in R$.
\[ \therefore \] $R$ is transitive.

Only if part: (If $R$ is transitive, then $\forall n \ R^n \subseteq R$) Use induction on $n$.
Basis step: $R^1 \subseteq R$; true for $n = 1$.
Inductive step: Assume $R^n \subseteq R$ and $R$ is transitive. Show $R^{n+1} \subseteq R$.

Let $(a, b) \in R^{n+1} = R^n \circ R$.
Then $\exists x \in A$ s.t. $(a, x) \in R$ and $(x, b) \in R^n$. Since $R^n \subseteq R$, $(x, b) \in R$.
Since $R$ is transitive and $(a, x) \in R$, we have $(a, b) \in R$
\[ \therefore \] $R^{n+1} \subseteq R$
**Inverse and Complementary:**

**Inverse** of \( R \): \( R^{-1} = \{(b, a) \mid (a, b) \in R\} \)

**Complementary of** \( R \): \( \overline{R} = \{(a, b) \mid (a, b) \notin R\} \)
Inverse and Complementary:

**Inverse** of $R$: $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

**Complementary** of $R$: $\bar{R} = \{(a, b) \mid (a, b) \notin R\}$

e.g.

Let $R = \{(a, b) \mid a < b\}$ $R: A \to B$.

Inverse of $R$: $R^{-1} = \{(b, a) \mid a < b\}$
Complementary of $R$: $\bar{R} = \{(a, b) \mid a \geq b\}$
e.g. 
$R, S$ are reflexive relations on $A$.

a) $R \cup S$ is reflexive? Yes, since $(x, x) \in R$
b) $R \cap S$ is reflexive?  

\checkmark  

so does $R \cup S$
c) $R \oplus S$ is irreflexive?  

\checkmark
d) $R - S$ is irreflexive?  

\checkmark
e) $S \circ R$ is reflexive?  

\checkmark
f) $R^{-1}$ is reflexive?
g) Complementary of $R$ is irreflexive?
e.g.
Suppose $R$ is **irreflexive**. Is $R^2$ also irreflexive?
No. Counter-example: Let $a \neq b$ and $R = \{(a, b), (b, a)\}$
9.2 \( n \)-ary Relations and Their Applications

*Definition:*
Let \( A_1, A_2, \ldots, A_n \) be sets.
An \( n \)-ary relation on these sets is a subset of \( A_1 \times A_2 \times \ldots \times A_n \).

The sets \( A_i \): Domains of the relation
\( n \): Degree of the relation

e.g.
\[
R = \{(a, b, c) \mid a < b < c\}
\]
**Databases and Relations**

The way we organize information in a database is important. Operations such as add/delete record, update records, search for record, all have heavy computation.

∴ Various methods for representing databases exist.

One method in particular is relational data model.

A database consists of records of $n$-tuples, made up of domains (fields).

*e.g.* Airflight Company  (Flight No, Departure, Destination, Date)

You will have an elective database course in 3\textsuperscript{rd} or 4\textsuperscript{th} year.
9.3 Representing Relations

**Definition:** A relation $R$ can also be represented by a matrix $M_R = [m_{ij}]$:

$$m_{ij} = \begin{cases} 
1 & \text{if } (a_i, b_j) \in R \\
0 & \text{if } (a_i, b_j) \notin R 
\end{cases}$$

E.g. Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and $R: A \rightarrow B$ such that $R = \{(1, b), (2, a), (2, b), (2, c)\}$

$$M_R = \begin{bmatrix} 
0 & 1 & 0 \\
1 & 1 & 1 
\end{bmatrix}$$
9.3 Representing Relations

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m_{ij} = \begin{cases} 
1 & \text{if } (a_i, b_j) \in R \\
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\end{cases}
\]

*E.g.* Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and $R: A \to B$ such that $R = \{(1, b), (2, a), (2, b), (2, c)\}

\[
M_R = \begin{bmatrix}
0 & 1 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

*E.g.* Let $R$ be a relation defined on $A = \{1, 2, 3\}$: $R = \{(1, 2), (2, 2), (1, 3)\}

\[
M_R = \begin{bmatrix}
0 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Note that we get a square matrix whenever $R: A \to A$. 

- Reflexive relation $R$ s.t. $(a_i, a_i) \in R$
  \[ \Rightarrow \forall i \; m_{ii} = 1 \]
  i.e., $M_R = \begin{bmatrix} 1 & & \cdots & 1 \\ & 1 & \cdots & \cdot \\ \vdots & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdots & 1 \end{bmatrix}$ diagonal with all ones

- Symmetric relation $R$ s.t. $(a_i, a_j) \in R \iff (a_j, a_i) \in R$
  \[ \Rightarrow \forall i, j \; m_{ij} = m_{ji} \]
  $M_R = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \cdot & \ddots & \cdot \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$ Symmetric matrix
  ($M_R = M^T_R$)
- Inverse and complementary relations:

If $M_R = [m_{ij}]_{m \times n}$, then

**Inverse:** $M_R^{-1} = [m_{ji}]_{n \times m}$ (transpose)

**Complementary:** $\overline{M_R} = [-m_{ij}]_{m \times n}$ (negation)
**Using Zero – One Matrices:**

A matrix with entries that are either 0 or 1 is called a **zero-one matrix**.

**Definition:**

\[
A = [a_{ij}], \quad B = [b_{ij}] \quad m \times n \text{ zero-one matrices}
\]

**Join** of A, B: \( A \lor B = [a_{ij} \lor b_{ij}] \)

**Meet** of A, B: \( A \land B = [a_{ij} \land b_{ij}] \)

*e.g.*

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix} \quad B = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
A \lor B = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix} \quad A \land B = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]
**Using Zero – One Matrices:**

A matrix with entries that are either 0 or 1 is called a **zero-one matrix**.

**Definition:**

\[ A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad B = \begin{bmatrix} b_{ij} \end{bmatrix} \quad m \times n \text{ zero-one matrices} \]

**Join of** \( A, B \): \( A \lor B = [a_{ij} \lor b_{ij}] \)

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e.g.

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix} \quad \quad B = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
A \lor B = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix} \quad A \land B = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

**Remark:** Let \( R_1: A \rightarrow B \) and \( R_2: A \rightarrow B \)

\[
\begin{align*}
M_{R_1 \cup R_2} &= M_{R_1} \lor M_{R_2} \\
M_{R_1 \cap R_2} &= M_{R_1} \land M_{R_2}
\end{align*}
\]
**Definition:**  **Boolean product**  
Let \( A = [a_{ij}] : m \times k \), \( B = [b_{ij}] : k \times n \) zero-one matrices

\[
A \odot B = [c_{ij}] : m \times n, \text{ where }
\]

\[
c_{ij} = (a_{i1} \wedge b_{1j}) \lor (a_{i2} \wedge b_{2j}) \lor \ldots \lor (a_{ik} \wedge b_{kj})
\]
e.g.

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}_{3 \times 2}
\]

\[
B = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}_{2 \times 3}
\]

\[
A \oplus B = \begin{bmatrix}
(1 \land 1) \lor (0 \land 0) & (1 \land 1) \lor (0 \land 1) & (1 \land 0) \lor (0 \land 1) \\
(0 \land 1) \lor (1 \land 0) & (0 \land 1) \lor (1 \land 1) & (0 \land 0) \lor (1 \land 1) \\
(1 \land 1) \lor (0 \land 0) & (1 \land 1) \lor (0 \land 1) & (1 \land 0) \lor (0 \land 1)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}_{3 \times 3}
\]
e.g.

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix}_{3 \times 2} \quad B = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}_{2 \times 3}
\]

\[
A \odot B = \begin{bmatrix}
(1 \land 1) \lor (0 \land 0) & (1 \land 1) \lor (0 \land 1) & (1 \land 0) \lor (0 \land 1) \\
(0 \land 1) \lor (1 \land 0) & (0 \land 1) \lor (1 \land 1) & (0 \land 0) \lor (1 \land 1) \\
(1 \land 1) \lor (0 \land 0) & (1 \land 1) \lor (0 \land 1) & (1 \land 0) \lor (0 \land 1)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}_{3 \times 3}
\]

**Remark:** Let $R: A \rightarrow B$ and $S: B \rightarrow C$

\[
M_{S \circ R} = M_R \odot M_S
\]
**Definition:** $r$ th **Boolean Power**
Let $A$ be a square $(n \times n)$ zero-one matrix and $r$ be a positive integer.

$$A^r = A \odot A \odot \ldots \odot A$$
$r$ times

$$A^0 = I_n$$

**Remark:** Let $R: A \rightarrow A$

$$M_{R^a} = [M_R]^n$$
Representing Relations Using Graphs:

Pictorial representation.

Definition:
A directed graph (digraph) consists of a set $V$ of vertices (or nodes) along with a set $E$ of edges (or arcs) which are ordered pairs of vertices.

Edge$(a, b)$: $a$ is initial vertex (node), $b$ is terminal vertex (node)

e.g.

\[ R = \{(a, b), (b, c), (c, b), (c, c)\} \]

Relation $R$ on a set $A$ is defined with
i) elements of $A$: vertices (nodes)
ii) ordered pairs $(a, b)\in R$: edges
Relation $R$ is:
- reflexive  iff every node has a loop
- symmetric  iff every edge between two nodes has an edge in the opposite direction.
- transitive  iff edge $(a, b) \land \text{edge} (b, c) \rightarrow \text{edge} (a, c) \ \forall a,b,c$

\[ e.g. \]

\[ \text{reflexive} \]
Example to graph representation of a relation:

Connectivity problems:

1) Which nodes are connected?
2) What is the shortest path between two nodes?
9.4 Closures of Relations

e.g. Let $R = \{(1, 1), (1, 2), (3, 2)\}$ on $A = \{1, 2, 3\}$

$R$ is not reflexive; what is the smallest possible reflexive relation containing $R$?
9.4 Closures of Relations

e.g. Let $R = \{(1,1), (1, 2), (3, 2)\}$ on $A = \{1, 2, 3\}$

$R$ is not reflexive; what is the smallest possible reflexive relation containing $R$?

$$S = \{(1, 1), (1, 2), (3, 2), (2, 2), (3, 3)\}$$

$S$ is the reflexive closure of $R$. 
**Definition: Closure**
Let $R$ be a relation on $A$

$P$: some property, such as symmetry, reflexivity, transitivity

$R$ may or may not have the property $P$.

The **closure** $S$ is the smallest possible set with property $P$, which contains $R$. 
**Definition: Closure**
Let $R$ be a relation on $A$

$P$: some property, such as symmetry, reflexivity, transitivity

$R$ may or may not have the property $P$.

The **closure** $S$ is the smallest possible set with property $P$, which contains $R$.

More formal definition of **closure**:

If there is a relation $S$ with property $P$ containing $R$ s.t. $S$ is the subset of every relation with property $P$ containing $R$, then $S$ is called the **closure** of $R$ with $P$. 
**Reflexive Closure:**

Let \( R = \{(1,1), (1, 2), (3, 2)\} \) on \( A = \{1, 2, 3\} \)

The smallest possible reflexive relation containing \( R \):

\[
S = \{(1, 1), (1, 2), (3, 2), (2, 2), (3, 3)\}
\]

\( S = \text{Reflexive closure of } R = R \cup \Delta, \)
where \( \Delta = \{(a, a) \mid a \in A\} \) : diagonal relation
**Reflexive Closure:**

Let $R = \{(1,1), (1, 2), (3, 2)\}$ on $A = \{1, 2, 3\}$

The smallest possible reflexive relation containing $R$:

$$S = \{(1, 1), (1, 2), (3, 2), (2, 2), (3, 3)\}$$

$S = \text{Reflexive closure of } R = R \cup \Delta,$

where $\Delta = \{(a, a) \mid a \in A\}$ : diagonal relation

e.g.

$$R = \{(a, b) \mid a < b\}, \quad \text{reflexive closure?}$$

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbb{Z}\}$$

$$= \{(a, b) \mid a \leq b\}$$
**Symmetric Closure:**

Let $R = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1), (3, 3)\}$ on $A = \{1, 2, 3\}$

We should add all ordered pairs $(b,a)$, where $(a, b)$ is in $R$ and $(b, a)$ is not in $R$. Symmetric closure of $R = R \cup \{(3, 2), (1, 3)\}$
**Symmetric Closure:**

Let \( R = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1), (3, 3)\} \) on \( A = \{1, 2, 3\} \)

We should add all ordered pairs \((b,a)\), where \((a, b)\) is in \( R \) and \((b, a)\) is not in \( R \).
Symmetric closure of \( R = R \cup \{ (3, 2), (1, 3) \} \)

Symmetric closure of \( R = R \cup R^{-1} \)  (since \( R^{-1} = \{ (b, a) \mid (a, b) \in R \} \))

*e.g.*
\( R = \{(a, b) \mid a < b\} \)

Symmetric closure of \( R = R \cup R^{-1} \)
\[
= \{(a, b) \mid a < b\} \cup \{ (b, a) \mid a < b\}
= \{(a, b) \mid a \neq b\}
\]
Transitive Closure:

Let \( R = \{(1, 3), (1, 4), (2, 1), (3, 2)\} \) on \( \{1, 2, 3, 4\} \)

\( R \) is not transitive since there are pairs \((a, c) \notin R\) although \((a, b), (b, c) \in R\).

(i) \( R \cup \{(1, 2), (2, 3), (2, 4), (3, 1)\} \)

Is it transitive?
**Transitive Closure:**

Let $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on $\{1, 2, 3, 4\}$

$R$ is not transitive since there are pairs $(a, c) \not\in R$ although $(a, b), (b, c) \in R.$

(i) $R \cup \{(1, 2), (2, 3), (2, 4), (3, 1)\}$

Is it transitive? NO!

It has $(3, 1), (1, 4),$ but not $(3, 4).$

We have a more difficult problem!!

We might repeat step (i) until reaching a transitive relation. But there are better ways.
e.g. Draw reflexive closure of

![Diagram showing reflexive closure]

How about symmetric closure? Transitive closure?
**Paths in Directed Graphs**

We now introduce a new terminology that we will use in the construction of transitive closures.

*Definition:*

A path from $a$ to $b$ in the directed graph $G$ is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in $G$ where $x_0 = a$ and $x_n = b$. This path is denoted by $x_0, x_1, \ldots, x_n$ and has a length of $n$.

If $x_0 = x_n$, the path is called a cycle or circuit.

Two vertices are said to be connected if there’s a path between them.

*e.g.*

A path: $a, b, d, a, c$

$a$ is connected to $e$, but $e$ is not connected to $a$.

The term path also applies to relations.
**Theorem:**
Let $R$ be a relation on $A$, then there is a path of length $n$ from $a$ to $b$ iff $(a, b) \in R^n$.

e.g.

```
A path:
  a, b, d, e
```

$(a, e) \in R^3$ since there is a path of length 3 between $a$ and $e$.  

![Diagram](image.png)
**Theorem:**
Let $R$ be a relation on $A$, then there is a path of length $n$ from $a$ to $b$ iff $(a, b) \in R^n$.

e.g.

\[
\begin{array}{c}
\text{A path:} \\
a, b, d, e
\end{array}
\]

$(a, e) \in R^3$ since there is a path of length 3 between $a$ and $e$.

But also $(a, e) \in R^6$ since there is also another path of length 6 between $a$ and $e$: $a, b, d, a, c, d, e$
Theorem:
Let $R$ be a relation on $A$, then there is a path of length $n$ from $a$ to $b$ iff $(a, b) \in R^n$.

Proof: Use induction.

Basis step:
By definition there is a path of length 1 from $a$ to $b$ iff $(a, b) \in R$. Hence true for $n = 1$. 
**Theorem:**
Let $R$ be a relation on $A$, then there is a path of length $n$ from $a$ to $b$ iff $(a, b) \in R^n$.

**Proof:** Use induction.

**Basis step:**
By definition there is a path of length 1 from $a$ to $b$ iff $(a, b) \in R$. Hence true for $n = 1$.

**Inductive step:** Assume it is true for some arbitrary fixed $n$. Show for $n+1$.

There is a path of length $n+1$ from $a$ to $b$ iff

$$\exists c \in A \text{ s. t. there is a path of length 1 from } a \text{ to } c \text{ and a path of length } n \text{ from } c \text{ to } b$$
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that is, $\exists c \in A$ such that $(a, c) \in R$ and $(c, b) \in R^n$ (by inductive hypothesis)
**Theorem:**
Let $R$ be a relation on $A$, then there is a path of length $n$ from $a$ to $b$ iff $(a, b) \in R^n$.

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**Basis step:**
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There is a path of length $n+1$ from $a$ to $b$ iff

$$\exists c \in A \text{ s. t. there is a path of length 1 from } a \text{ to } c \text{ and a path of length } n \text{ from } c \text{ to } b$$

that is, $\exists c \in A$ such that $(a, c) \in R$ and $(c, b) \in R^n$ (by inductive hypothesis)

which implies $(a, b) \in R^{n+1}$ (by definition of composite relation).

\[\therefore \text{ There is a path of length } n + 1 \text{ from } a \text{ to } b \text{ iff } (a, b) \in R^{n+1}\]
Transitive Closure:

Finding transitive closure is equivalent to determining vertices that are connected by a path.

Definition:
Let $R$ be a relation on $A$.
Connectivity relation $R^*$ consists of all pairs $(a, b)$ s.t. there’s a path between $a$ and $b$ in $R$.

Since $R^n$ includes all the paths of length $n$ by the above theorem,

$$R^* = \bigcup_{n=1}^{\infty} R^n$$
**Transitive Closure:**

Finding transitive closure is equivalent to determining vertices that are **connected** by a path.

*Definition:*
Let $R$ be a relation on $A$.
**Connectivity relation** $R^*$ consists of all pairs $(a, b)$ s.t. there’s a path between $a$ and $b$ in $R$.

Since $R^n$ includes all the paths of length $n$ by the above theorem, 

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

* e.g.
Let $R$ be a relation on the set of people in the world that contains $(a,b)$ if $a$ has met $b$.

$R^2$: ? if $(a, b) \in R^2$ then $\exists c$ s.t. $(a, c) \in R$ and $(c, b) \in R$

$R^*$: ? $(a, b) \in R^*$ if there is a sequence of people, starting with $a$ and ending with $b$. 
Theorem:
The transitive closure of a relation $R$ equals to the connectivity relation $R^*$. 
**Theorem:**
The transitive closure of a relation $R$ equals to the connectivity relation $R^*$.  

**Proof:**
We must show that, (i) $R^*$ is transitive and (ii) any transitive relation that contains $R$ contains also $R^*$.  

**Theorem:**
The transitive closure of a relation $R$ equals to the connectivity relation $R^*$.  

**Proof:**
We must show that, (i) $R^*$ is transitive and (ii) any transitive relation that contains $R$ contains also $R^*$.  

**i.** $R^*$ is transitive?  
If $(a, b) \in R^*$, there is a path from $a$ to $b$.  
If $(b, c) \in R^*$, there is a path from $b$ to $c$.  
\therefore There is a path from from $a$ to $c$, which means $(a, c) \in R^*$.  

**ii.**
**Theorem:**
The transitive closure of a relation $R$ equals to the connectivity relation $R^*$.

**Proof:**
We must show that, (i) $R^*$ is transitive and (ii) any transitive relation that contains $R$ contains also $R^*$.

**i.** $R^*$ is transitive?
If $(a, b) \in R^*$, there is a path from $a$ to $b$.
If $(b, c) \in R^*$, there is a path from $b$ to $c$.
$\therefore$ There is a path from $a$ to $c$, which means $(a, c) \in R^*$.

**ii.** Let $S$ be any transitive relation that contains $R$, i.e. $R \subseteq S$. Show $R^* \subseteq S$. 
Theorem:
The transitive closure of a relation $R$ equals to the connectivity relation $R^*$.  
Proof:
We must show that, (i) $R^*$ is transitive and (ii) any transitive relation that contains $R$ contains also $R^*$. 

i. $R^*$ is transitive?
If $(a, b) \in R^*$, there is a path from $a$ to $b$.
If $(b, c) \in R^*$, there is a path from $b$ to $c$.
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ii. Let $S$ be any transitive relation that contains $R$, i.e. $R \subseteq S$. Show $R^* \subseteq S$.
Since $S$ is transitive, $S^n \subseteq S$ (by the theorem in Sec. 9.1)

$S^n \subseteq S$ and $S^* = \bigcup_{n=1}^{\infty} S^n \Rightarrow S^* \subseteq S$
\Rightarrow since $R \subseteq S$ (given), $R^* \subseteq S^* \subseteq S \Rightarrow R^* \subseteq S$.
Thus any transitive relation $S$ that contains $R$ contains also $R^*$. 

66
Given $R$, how can we compute the connectivity relation $R^*$?

$$R^* = \bigcup_{n=1}^{\infty} R^n$$
Given $R$, how can we compute the connectivity relation $R^*$?

**Lemma:**
Let $R$ be a relation in $A$ and $|A| = n$. If there is a path from $a$ to $b$ in $R$, then one can always find a path from $a$ to $b$ with length not exceeding $n$. 
Given \( R \), how can we compute the connectivity relation \( R^* \)?

**Lemma:**
Let \( R \) be a relation in \( A \) and \( |A| = n \). If there is a path from \( a \) to \( b \) in \( R \), then one can always find a path from \( a \) to \( b \) with length not exceeding \( n \).

**Proof:**
Suppose there is a path \( x_0, x_1, \ldots, x_m \) from \( x_0 = a \) to \( x_m = b \) with length \( m \).
If \( m > n \), then there are at least two vertices on this path, equal to each other \( x_i = x_j \) such that \( 0 \leq i < j \leq m - 1 \). (by the pigeonhole principle)

We can cut this circuit and form a new path
\[
x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_m
\]
If we do the same for all such two vertices, we get a path of length \( \leq n \).
Given $R$, how can we compute the connectivity relation $R^*$?

**Lemma:**
Let $R$ be a relation in $A$ and $|A| = n$. If there is a path from $a$ to $b$ in $R$, then one can always find a path from $a$ to $b$ with length not exceeding $n$.

Hence by the Lemma,

$$R^* = \bigcup_{k=1}^{\infty} R^k = \bigcup_{k=1}^{n} R^k$$