1. Let $P(n)$: If $p$ is prime and $p|a_1a_2\cdots a_n$, where $a_i$ is an integer for $i = 1, 2, 3, \ldots, n$, then $p|a_i$ for some $i$. So we are asked to prove $\forall n \in \mathbb{Z}^+ P(n)$. Let's start the induction from $n = 2$.

Basis step: $P(2)$: If $p$ is prime and $p|a_1a_2$, then $p|a_1 \lor p|a_2$. To show that $P(2)$ is T, we reason in the following way. If $p$ is prime, then $p$ divides $a_1$ or $\gcd(p, a_1) = 1$. Also if $p|a_1a_2$ and $\gcd(p, a_1) = 1$, then by Lemma 1 of Chapter 4.3, $p|a_2$. Hence $p|a_1 \lor p|a_2$. Thus $P(2)$ is T.

Inductive Step: We need to show that $P(n) \rightarrow P(n+1)$, that is, by assuming that the inductive hypothesis $P(n)$ is true (for an arbitrary fixed $n$), we have to show that $P(n+1)$ is also true. $P(n+1)$ can be written as

$P(n+1)$: If $p$ is prime and $p|a_1a_2\cdots a_na_{n+1}$, then $p|a_i$ for some $i = 1, 2, \ldots, n+1$.

We proceed as follows:

$p|a_1a_2\cdots a_na_{n+1} \Rightarrow p|(a_1a_2\cdots a_n)a_{n+1}$
$\Rightarrow p|a_1a_2\cdots a_n \lor p|a_{n+1}$ by basis step
$\Rightarrow p|a_i$ for some $i = 1, 2, \ldots, n \lor p|a_{n+1}$ since $P(n)$ is true by inductive hypothesis
$\Rightarrow p|a_i$ for some $i = 1, 2, \ldots, n, n+1$.

Hence $P(n+1)$ is T whenever $P(n)$ is T. Hence by induction $\forall n \leq 2 \ P(n)$.

Remark: We could start the induction from $n = 1$ as well since $P(1)$ is clearly T. However we have chosen to start from $n = 2$, since we use the basis step later in the inductive step. Hence actually $P(n)$ is true for all positive $n$ as stated in the original lemma.

2. We prove this by induction on $n$. The base case is $n = 1$, and $f_1^2 = f_1f_2 = 1$, so the equation holds for the base case. Now, we assume the inductive hypothesis, that is, we assume that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_nf_{n+1}$. Adding $f_{n+1}^2$ to both sides, we get $f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 = f_nf_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1}f_{n+2}$, as desired. This concludes the proof.
3. We are asked to prove, \( \forall n \geq 0, f_n < (5/3)^n \), where is \( f_n \) the \( n \)th Fibonacci number, i.e., \( f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 \) and \( f_0 = 0, f_1 = 1 \). We will use strong induction.

**Basis step:** For \( n = 0 \), we have \( f_0 = 0 = (5/3)^0 = 1 \).

For \( n = 1 \), we have \( f_1 = 1 < (5/3)^1 = 5/3 \).

Note that at the basis step we need to show both for \( n = 0 \) and \( n = 1 \) since the value of the \( n \)th Fibonacci number \( f_n \) depends on two previous values.

**Inductive step:** We need to show that \( (\forall k \leq n, f_k < (5/3)^n) \rightarrow f_{n+1} < (5/3)^{n+1} \), for \( n \geq 2 \).

We proceed as follows:

\[
\begin{align*}
  f_{n+1} &= f_n + f_{n-1} < (5/3)^n + (5/3)^{n-1} \quad \text{(by inductive hypothesis)} \\
  &= (1 + 5/3) (5/3)^{n-1} < (5/3) \cdot (5/3)^{n-1} = (5/3)^{n+1} \quad \text{(since } 1 + 5/3 < (5/3)\text{)}
\end{align*}
\]

Thus, by strong induction, \( \forall n \geq 0, f_n < (5/3)^n \).